Simulation of "Impulse" Function $\hat{d}(x)$ of Min-Plus Convolution

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Abstract Authors regard discrete impulse $\hat{d}(x)$ in min-plus convolution as an impulse train which is a periodic function in the domain of generalized function. As for simulation, such an infinite series may be approached by a finite one and the error caused by truncating can be got rid of by a threshold selected. The infinite magnitude of $\hat{d}(x)$ may be simulated by the way of multiplying the simulation result by a constant so that the product is large enough in comparison with the signals convoluted or simply take the largest number in the computation software as an infinity. We illustrate these techniques and the convergence of the discrete $\hat{d}(x)$ in min-plus convolution in this paper. The fact that the discrete $\hat{d}(x)$ approaches its continuous case when its period approaches zero is also discussed.

Keywords Min-plus convolution, network calculus, Dirac-δ function, generalized functions

1. Introduction

To model networks in deterministic way, service curve based methods are attractive [1-4]. In this field, min-plus algebra is a tool of establishing a relationship between an input and an output of a network element. An important operation in min-plus algebra is convolution in the sense of min-plus. In min-plus convolution, there is an element called "impulse function" $\hat{d}(x)$ defined as $\hat{d}(x) = \begin{cases} \infty, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$. Clearly, $\hat{d}(x)$ is not a conventional function. Hence, come the difficulties in explanation of it and in doing its digital computation from the point of view of mathematics.

When applying min-plus to modeling networks practically, it is necessary to compute the operations related to the min-plus convolution. Our analysis results are not only meaningful for digital computation of $\hat{d}(x)$ in the field of min-plus convolution, but also useful for computation of impulse train in generalized functions as well as signal and linear system analysis.

We briefly introduce the concept of min-plus and point out the difficulties in digital computation of $\hat{d}(x)$ in section 2, expresses it as a Fourier series in the domain of generalized functions in section 3. In section 4, we illustrate the simulations of discrete $\hat{d}(x)$ by approximation with a finite series, explain how to eliminate the errors caused by truncating. Section 5 shows the convergence of $\hat{d}(x)$ of min-plus. Section 6 illustrates the simulation of continuous $\hat{d}(x)$.

2. Concept of Min-Plus Convolution and Its Impulse Function

For two functions $f_1(x)$, $f_2(x) \in \mathbf{R}$ ($\mathbf{R}$ is the set of real number), the operation of min-plus convolution is defined as

$$f_1(x)*f_2(x) = \min_{0\le u\le x}\{f_1(u)+f_2(x-u)\} \quad (1)$$

where, $*$ stands for the operation on the right hand of equation (1). Abstractly, it is similar to the operation of conventional convolution [7]. The operation for $f_1(x)$ and $f_2(x)$ under equation (1) is called min-plus convolution. In network calculus, the function

$$\hat{d}(x)=\begin{cases}\infty, & x\ge 0\\ 0, & \text{otherwise}\end{cases} \quad (2)$$

is called "impulse" function. [1-4] described its functionality by the following expression.

$$\hat{d}(x)*f(x) = f(x), \quad \forall f(x)\in R \quad (3)$$

According to (1) and (2), the operation of Equ. (3) seems natural. However, the question from Equ. (3) is sharp. As $\infty \notin \mathbf{R}$, how to do the operation with Equ. (3) as Equ. (1) naturally requires both functions $f_1(x)$ and $f_2(x) \in \mathbf{R}$?.

3. Discrete $\hat{d}(x)$ and its Fourier series

Assume that $x\in I^+$ (I^+ is a set of positive integer including zero), then $f(x) \in \mathbf{R}$ is a discrete function and so is $\hat{d}(x)$. For simplicity, we take $x = 0, T, 2T,\cdots, nT,\cdots$, where T is the sampling period of the function. Consequently, "impulse" function $\hat{d}(x)$ is a periodic function called impulse train expressed as

$$\hat{d}(x)=\begin{cases}\infty, & x=0,T,2T,\cdots\\ 0, & \text{otherwise}\end{cases} = \sum_{n=0}^{\infty} d(x-nT) \quad (4)$$

where δ(x) is Dirac-δ function. Denote impulse train $d_T(x)$ as

$$\sum_{n=0}^{\infty} d(x-nT) = d_T(x) \quad (5)$$

and it is obviously a generalized function [6]. Hence, the discrete "impulse" $\hat{d}(x)$ has its explanation in generalized functions.

Even in the case of discrete, one may still meet the difficulties in computation of it digitally. The difficulties are mainly from two aspects.

- Discrete function $\hat{d}(x)$ is an infinite series. It is divergence in the domain of conventional functions.
- The values of discrete $\hat{d}(x)$ are infinite.

Two difficulties above can be smoothed away in the following way. The first difficulty can be overcome by truncating in the sense of approximation. The second can be approximated by expanding it into a Fourier series in the domain of generalized function below [6, 7]

$$\hat{d}(x)=\sum_{n=0}^{\infty} d(x-nT) = \frac{2}{T}+\frac{4}{T}\sum_{n=1}^{\infty}\cos\left(\frac{2npx}{T}\right) \quad (6)$$

Equ. (6) is the formula for simulation of discrete "impulse" $\hat{d}(x)$. As mentioned, we need approximation with a finite series as

$$\hat{d}(x)\approx\frac{2}{T}+\frac{4}{T}\sum_{n=1}^{N}\cos\left(\frac{2npx}{T}\right) \quad (7)$$

The error caused is

$$error(x;N)=\frac{4}{T}\sum_{n=N+1}^{\infty}\cos\left(\frac{2npx}{T}\right) \quad (8)$$

Therefore, the simulation should handle:

- How to eliminate the error caused by approximation in practice;
- How to make the magnitude of impulse "infinite".

The next section will show the techniques related.

4. Simulation of Discrete Impulse

In this section, we focus on the illustrations to explain the techniques of simulation.

Finite Approximation

Consider Equ. (7) under $N = 10$, $T = 9$, $x = 0 \sim 300$. The computation result is shown as Fig. 1

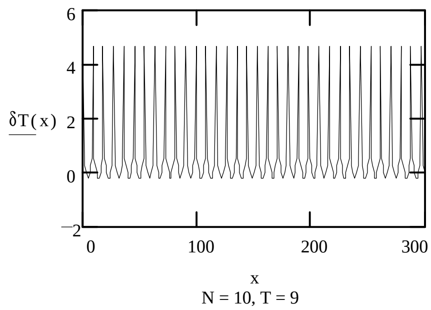


Fig. 1 Finite approximation with error

The components among each two conjoint *xs* are caused by finite approximation. They are error components. Therefore, the way to eliminate error components in practice is to set a threshold to cut them away in a computation program.

Elimination of Error

When **threshold = 2** in this example, the result of approximation is as Fig. 2 shows

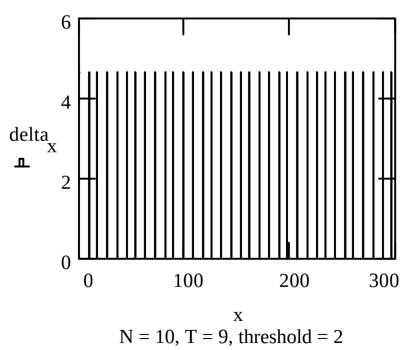


Fig. 2 Finite approximation without error

Thus, in the range of $x = 0 \sim 300$, the function delta_x is exactly the same as an impulse train except that the magnitudes are not infinite. The following subsection will show how to solve the magnitude problem mentioned just now.

Control of Magnitude

To simulate an impulse train digitally, we may multiply a constant to the result delta_x in the program. In this way, one may take the largest number the computation software used to represent the infinity if needed. For instance, the largest number the software we used is 10^{307}, hence the result is shown as Fig. 3 although it is not necessary to do so, generally speaking. Practically, it is wise to set the magnitudes large enough in comparison with the functions one is going to convolute.

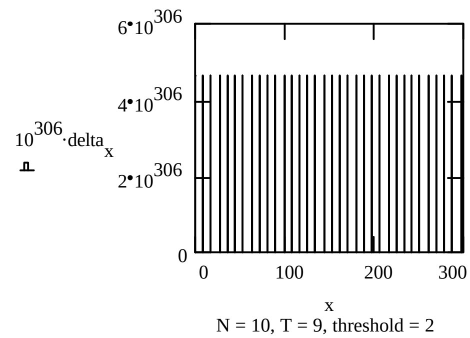
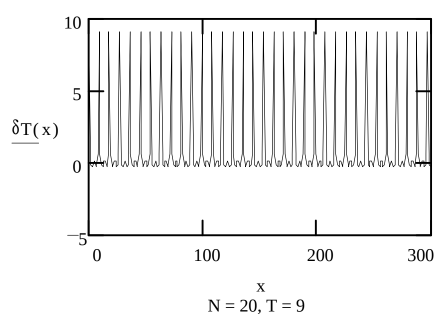


Fig. 3 Approximation with infinite magnitude

5. Convergence of Discrete Impulse

For simplicity, we illustrate the approximation results when $N \to \infty$. Fig. 4 shows the result when $N = 20, 40, 60$ respectively, which indicate that error components are decreased and the magnitudes are increased as N increases.



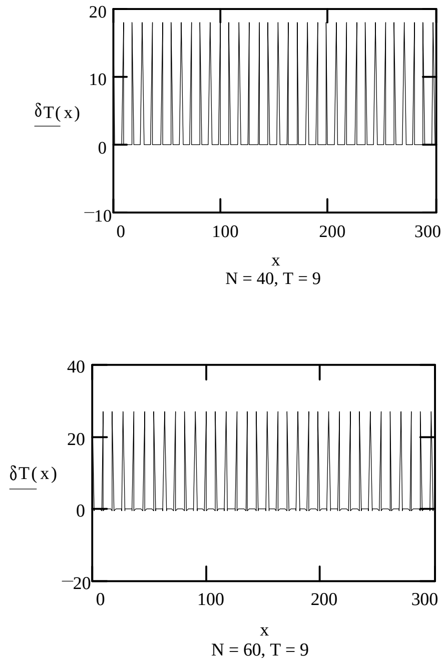


Fig. 4 Observation of convergence

6. From Discrete to Continuous

When the period T concerned is small enough in comparison with the sampling period of the signal to be convoluted, the impulse train approaches its continuous function $\hat{d}(x)$. That is,

$$d_T(x)\big|_{T\to 0} = \hat{d}(x) = \begin{cases} \infty, & x \geq 0 \\ 0, & \text{otherwise} \end{cases} \quad (9)$$

Practically, $T \to 0$ means the T is small enough

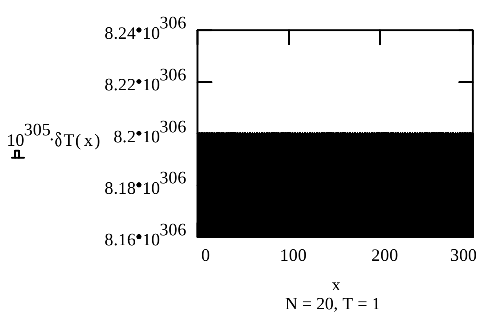


Fig. 5 Discrete impulse approaches its continuous case

compared to the sampling period of signal. Fig. 5 shows the result the case when $T = 1$.

7. Conclusions

Though the impulse train is an infinite series in math, it may be approximated by a finite series with a certain amount of error components which can be cut away by setting a threshold in computation program. The magnitude of the impulse can be simulated digitally by multiplying the approached result by a constant. It approaches its continuous case when the period approaches zero. These are techniques of the numerical simulation of it in min-plus convolution in packet switched networks.

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