Planar Graphs

Lecture 10: Oct 21
This Lecture

Today we will talk about planar graphs, and how to color a map using 6 colors.

• Planar graphs
• Euler’s formula
• 6-coloring
Map Colouring

Colour the map using minimum number of colours so that two countries sharing a border are assigned different colours.
Map Colouring

Can we draw a map so that there are 5 countries such that any two of which are adjacent? NO

Can we draw a map which needs 5 colours? NO

Conjecture (1852) Every map is 4-colourable.

“Proof” by Kempe 1879, an error is found 11 years later.

(Kempe 1879) Every map is 5-colourable.

Theorem (Apple Haken 1977). Every map is 4-colourable.

The proof is computer assisted, some mathematics are still not happy.
Planar Graphs

- Each vertex is a region.
- Two regions have an edge if they are adjacent.

This is a planar graph.

A graph is **planar** if there is a way to **draw** it in the plane without edges crossing.
Non-Planar Graphs

Can we draw a map so that there are 5 countries such that any two of which are adjacent? NO
An important concept of a planar graph is its faces. So let’s study it in some details.
Region Boundaries
Region Boundaries: Bridge
Region Boundaries: Bridge

abcefgecdad

outer region
Region Boundaries: Dongle
Region Boundaries: Dongle

outer region
Region Boundaries: Dongle
Region Boundaries: Dongle
Planar Embeddings

A **planar embedding** is a graph *along with* its face boundaries: cycles
(same graph may have different embeddings)

![Diagram](image)

- two length 5 faces
- length 7 face
This Lecture

- Planar graphs
- Euler’s formula
- 6-coloring
Euler's Formula

If a connected planar graph has $n$ vertices, $m$ edges, and $f$ faces, then

$$n - m + f = 2$$

$n=5, m=5, f=2$

$n=6, m=10, f=6$

$n=9, m=8, f=1$
Proof of Euler's Formula

If a connected planar graph has $n$ vertices, $m$ edges, and $f$ faces, then
\[ n - m + f = 2 \]

Proof by induction on the number of vertices.

Base case (n=1):

```
\begin{itemize}
  \item n=1
  \item f=m+1
\end{itemize}
```
Proof of Euler's Formula

If a connected planar graph has $n$ vertices, $m$ edges, and $f$ faces, then

$$n - m + f = 2$$

Induction step ($n>1$):

"contract" the red edge

$n' = n-1$, $m' = m-1$, $f' = f$

Number of faces is the same, although some faces get shorter.

By induction, $n' - m' + f' = 2$. This implies $n - m + f = 2$. 
This Lecture

• Planar graphs
• Euler’s formula
• 6-coloring
Proof Steps

Theorem. Every planar graph is 6-colorable.

The strategy is similar to that in the previous lecture: to find a low degree vertex

There are three steps in the proof.

1) Show that there are at most $3n-6$ edges in a simple planar graph.

2) Show that there is a vertex of degree 5.

3) Show that there is a 6-coloring.
Claim. If \( G \) is a simple planar graph with at least 3 vertices, then
\[
m \leq 3n - 6
\]
Let \( F_1, \ldots, F_f \) be the face lengths.

Note that
\[
2m = \sum_{i=1}^{f} F_i
\]
because each edge contributes 2 to the sum.
Claim. If $G$ is a simple planar graph with at least 3 vertices, then
\[ m \leq 3n - 6 \]

Let $F_1, \ldots, F_f$ be the face lengths.

Note that $2m = \sum_{i=1}^{f} F_i$

Since the graph is simple, each face is of length at least 3.

So $2m = \sum_{i=1}^{f} F_i \geq 3f$

Since $m = n + f - 2$, this implies

\[ m \leq n + 2m/3 - 2 \quad \Rightarrow \quad m/3 \leq n - 2 \quad \Rightarrow \quad m \leq 3n - 6 \]
Claim. If $G$ is a simple planar graph with at least 3 vertices, then $m \leq 3n - 6$

Claim. Every simple planar graph has a vertex of degree at most 5.

1. Suppose every vertex has degree at least 6.
2. Then $m \geq 6n/2 = 3n$.
3. A contradiction.
**Claim.** Every simple planar graph has a vertex of degree at most 5.

**Theorem.** Every planar graph is 6-colorable.

1. Proof by induction on the number of vertices.
2. Let $v$ be a vertex of degree at most 5.
3. Remove $v$ from the planar graph $G$.
4. Note that $G-v$ is still a planar graph.
5. By induction, $G-v$ is 6-colourable.
6. Since $v$ has at most 5 neighbours,
7. $v$ can always choose a colour (from the 6 colours).
Application of Euler’s Formula

Can we draw a map so that there are 5 countries such that any two of which are adjacent?  NO

Can this graph have a planar drawing?

Claim. If $G$ is a simple planar graph with at least 3 vertices, then $m \leq 3n - 6$

This graph has $n=5$ and $m=10$, and so does not satisfy the claim.
Polyhedra (Optional)

Icosahedron gives a 5 regular planar graph.

So this approach cannot prove that every planar graph is 5-colorable.
Summary

It is not very difficult to prove that every planar graph is 5-colorable. (See wiki if you are interested.)

We have finished our second topic of this course, graph theory.

In this topic, I hope you learn
(1) how to apply the proof techniques in proving results in graph theory.
(2) how to model problems as graph problems
(3) how to reduce one problem to another
   (e.g. maximum matching to perfect matching, k-stroke graph to 1-stroke graph, etc.)

Reductions and inductions are important techniques in computer science.