Invariant and Graph

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Given a 4 by 4 matrix

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 \\
\end{bmatrix}
\]

You can only do 2 type of operations
1. Choose a row and flip it.
2. Choose a column and flip it.
Flip means change 0 to 1 and 1 to 0.

Show that you can not make a all 1 matrix only using the above operations.
Examples of operation:
1. Choose row 2 and flip it

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1
\end{bmatrix}
\]

2. Choose column 3 and flip it

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1
\end{bmatrix}
\]
Let $C$ and $R$ be two 4 dimensional vector. $C_i$ is the parity of column $i$ and $R_i$ is the parity of row $i$.

\[
\begin{array}{cccc}
0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 \\
\end{array}
\]

Originally, $R = \{0, 1, 1, 1\}$, $C = \{0, 1, 1, 1\}$.

Invariant: Every operation is either flipping $R$ or $C$. 
1. Choose row 2 and flip it.

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2. Choose column 3 and flip it.

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In this tutorial, we will consider simple graph only.
What is a simple graph?

T/F question
A graph without multi-edge is simple.
T/F question
A graph without multi-edge is simple.

Solution
False.
Simple graph should have no multi-edge and no self loop.
T/F question

3, 4, 3, 2, 1 is a degree sequence of a graph.
T/F question
3, 4, 3, 2, 1 is a degree sequence of a graph.

Solution
False.
Because sum of degree = 13 which is odd and violates hand shaking lemma.
question

What is the graph with every vertex has degree at most 2?
question
What is the graph with every vertex has degree at most 2?

Solution
The graph is formed by lines and cycles.
What is the maximum number of edges of a connected graph?
What is the maximum number of edges of a connected graph?

Solution

\[ \frac{n(n-1)}{2} \]

Every node connects to every other \((n - 1)\) nodes.

Sum of degree = \(n(n - 1)\).

Number of edges = \(\frac{\text{sum of degree}}{2} = \frac{n(n-1)}{2} \)

This kind of graph is called complete graph.
What is the minimum number of edges of a connected graph?
question
What is the minimum number of edges of a connected graph?

Solution
$n - 1$.
If the graph has a cycle, we can remove 1 edge from the cycle and it is still connected.
A graph doesn’t have cycle is a tree and has $n - 1$ edges.
You can not remove more edges from a tree otherwise it will be disconnected.
question
Show that a connected graph with \( n \) nodes and \( n - 1 \) edges is a tree.
question

Show that a connected graph with \( n \) nodes and \( n - 1 \) edges is a tree.

Solution

We can prove by induction. It is true when there is 1 or 2 nodes.

\[
\begin{align*}
\text{n = 1} & \quad \text{n = 2} \\
\begin{array}{c}
A \\
\end{array} & \quad \begin{array}{c}
A \\
\end{array} \\
& \quad \begin{array}{c}
B \\
\end{array}
\end{align*}
\]
question
Show that a connected graph with $n$ nodes and $n - 1$ edges is a tree.

Solution
Assume it is true when there are $k \geq 1$ nodes. When there are $k + 1$ nodes, call this graph $G$. There is at least a node $u$ with degree 1. And $u$ is connected to some node $v$. 

![Diagram](image-url)
Show that a connected graph with $n$ nodes and $n - 1$ edges is a tree.

Solution

Then if we remove $u$ and edge $(u, v)$, the remaining graph $G'$ has $k$ nodes and $k - 1$ edges.

By assumption, $G'$ is a tree. And we can see that adding edge $(u, v)$ and node $u$ to $G'$ doesn't create cycle. So $G$ is a tree.
T/F question

A graph with $n$ nodes and $k$ trees has $n - k$ edges.
T/F question
A forest with $n$ nodes and $k$ trees has $n - k$ edges.

Solution
True.
Let $V_i$ be number of nodes of the $i^{th}$ tree.
Let $E_i$ be number of edge of the $i^{th}$ tree.
Then $E_i = V_i - 1$
Total number of edges

$$= E_1 + E_2 + \ldots + E_k$$
$$= V_1 - 1 + V_2 - 1 + \ldots + V_k - 1$$
$$= n - k$$