Tutorial 10: Counting by mapping

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If $f$ is a bijection from $A$ to $B$, then $|A| = |B|$

It may be difficult to compute $|A|$ directly, so we define a new set/counting problem $B$

Prove that for each solution/element in $A$, it is equal to one and only one solution in $B$, and vice versa, i.e. bijection.

Compute $|B|$ and as it is bijection, we conclude that $|A| = |B|$
In tutorial 8...

Question
Starting in the bottom left corner, and we can only go up or right, how many routes are there to the top right corner in a 5 * 6 grid?

Solution
For each route, we map it to a string of length $n$ with 2 alphabet \{R, U\}
You are standing at position 0. Each step you move one step to right (+1) or left (−1).
Count the number of paths that:
a) Total number of steps is 6.
b) Total number of steps is 8 and ending point is +2
Question

You are standing at position 0. Each step you move one step to right \((+1)\) or left \((-1)\).
Count the number of paths that:
a) Total number of steps is 6.
b) Total number of steps is 8 and ending point is \(+2\)

Solution

Map the paths to 01-strings such that moving right is mapped to 0 and moving left is mapped to 1.
The length of string is the number of steps, and the final position determines the numbers of 0’s and 1’s.
a) \(2^6\)
b) 3 move to left and 5 move to right. \(\binom{8}{3}\)
Question

What is the number of non-negative integer solutions to
\[ x_1 + x_2 + \cdots + x_m = n? \]
Example: \( x_1 + x_2 + x_3 + x_4 = 20 \)
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**Solution**

- Like counting doughnut: 4 kinds of doughnuts, select 20
- Map it to a 23-bits string with 20 0s and 3 1s
- e.g. 000110000000001000000000 = 3 + 0 + 9 + 8
- \( \binom{23}{3} \)
- In general, \( n + m - 1 \) bits and \( m - 1 \) 1s to be "separator"
- \( \binom{n+m-1}{m-1} \)
Question

for i=1 to n do
    for j=1 to i-1 do
        for k=1 to j-1 do
            printf("hello world\n");

How many "hello world" are printed?
Question

```c
for i=1 to n do
    for j=1 to i-1 do
        for k=1 to j-1 do
            printf("hello world\n");
```

How many "hello world" are printed?

Solution

- Each "hello world" corresponds to a triple \((i, j, k)\) where \(n \geq i > j > k \geq 1\)
- Map to a \(n\)-bits string with exactly 3 1s
- The position of 1s represent the value of \(i, j, k\) respectively
- It is bijective since for each \((i, j, k)\), we can find one and only one corresponding string, and for each string we can find one and only one corresponding \((i, j, k)\)
- \(\binom{n}{3}\)
Question

What is the number of non-negative integer solutions to $x_1 + x_2 = n$ where $x_1 < a_1$ and $x_2 < a_2$?
E.g. $x_1 + x_2 = 10$ where $x_1 < 7$ and $x_2 < 8$
Question
What is the number of non-negative integer solutions to \( x_1 + x_2 = n \) where \( x_1 < a_1 \) and \( x_2 < a_2 \)?
E.g. \( x_1 + x_2 = 10 \) where \( x_1 < 7 \) and \( x_2 < 8 \)

Solution
- Inclusion-exclusion principle
- Total number of non-negative solutions: \( \binom{10 + 2 - 1}{1} = 11 \)
- Number of non-negative solutions with \( x_1 \geq 7 \): \( \binom{4}{1} = 4 \)
- Number of non-negative solutions with \( x_2 \geq 8 \): \( \binom{3}{1} = 3 \)
- Number of non-negative solutions with \( x_1 \geq 7 \) and \( x_2 \geq 8 \): 0
- Number of non-negative solutions with \( x_1 \geq 7 \) or \( x_2 \geq 8 \):
  \[ 4 + 3 - 0 = 7 \]
- Number of non-negative solutions with \( x_1 < 7 \) and \( x_2 < 8 \):
  \[ 11 - 7 = 4 \]
Solution

(Cont.)

In general:

- Total number of non-negative integer solutions: \( n + 1 \)
- Number of non-negative integer solutions with \( x_1 \geq a_1 \):
  \( n - a_1 \)
- Number of non-negative integer solutions with \( x_2 \geq a_2 \):
  \( n - a_2 \)
- Number of non-negative integer solutions with \( x_1 \geq a_1 \) and \( x_2 \geq a_2 \):
If a function from $A$ to $B$ is $k - to - 1$, meaning that each element in $B$ is mapped by exactly $k$ elements in $A$

$|A| = k|B|$

Example: 1 combination is mapped by $n!$ permutations

$n! \#\text{combination} = \#\text{permutation}$
Question

Suppose we want to divide $kn$ people into $n$ teams of $k$ people. How many possible combinations are there if the team names are important?
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Suppose we want to divide $kn$ people into $n$ teams of $k$ people. How many possible combinations are there if the team names are important?

Solution
- We can represent the team assignment by a string over the alphabet $\{1, 2, \cdots, n\}$
- If the $i$-th position of the string is $j$, then person $i$ is assigned to team $j$. Since there are exactly $k$ people in each team, every number from 1 to $n$ appears in exactly $k$ positions of such string.
- We are essentially asked to count the number of such strings
- L16 p.14, $(k!)^n$ to 1
- $\frac{(kn)!}{(k!)^n}$
Question
Suppose we want to divide $kn$ people into $n$ teams of $k$ people. How many possible combinations are there if the team names are **not** important?
Question
Suppose we want to divide $kn$ people into $n$ teams of $k$ people. How many possible combinations are there if the team names are not important?

Solution
- Let team name be $1^n$
- For every team assignment, we assign the team name by a $n$ permutation $n!$
- As the name is not important, those $n!$ permutation are the same assignment.
- $n!$ to 1 of the answer of previous question
- $\frac{(kn)!}{(k!)^{n}*n!}$
Question
A monotone path in $\mathbb{Z}^3$ from $(0, 0, 0)$ to $(n, n, n)$ is a path consists of ’x’-moves (this corresponds to a move that increases the x-coordinate by 1, while keeping the other coordinates the same), ’y’-moves and ’z’-moves, starting at $(0, 0, 0)$ and ending at $(n, n, n)$.

i.e. $(x, y, z) \rightarrow (x + 1, y, z)$ is valid, $(x, y, z) \rightarrow (x - 1, y, z)$ is not.
How many possible monotone paths are there from $(0, 0, 0)$ to $(n, n, n)$?
Solution

- Map to $3n$ string with $n-x, n-y, n-z$.
- For each string, we can change the position of $x/y/z$ ($n! to 1$).
- $(n!)^3$ to 1
- $\frac{(3n)!}{(n!)^3}$