Propositional Logic

1. **T/F** Which of the following are propositions?
   
   (a) \( x < 30 \).
   
   (b) I am hungry.
   
   (c) Open the windows.
   
   (d) \( 15 > 20 \).
   
   (e) Every even integer greater than 2 can be expressed as the sum of two primes.
   
   (f) \( a^2 + b^2 = c^2 \).
   
   (g) \( \pi + e \) is rational.

2. **SQ** Let \( h = \text{“Peter is handsome”, } c = \text{“Peter is clever”, } o = \text{“Peter is optimistic”}. \) Rewrite the following in symbolic forms using \( \neg, \lor, \land \). You don’t have to simplify the formulas.

   (a) Peter is handsome and clever but not optimistic.
   
   (b) Peter is either clever or handsome or both.
   
   (c) Peter is either clever or handsome but not both.
   
   (d) Peter is neither handsome, clever nor optimistic.
   
   (e) Peter is not both handsome and clever, but he is optimistic.
   
   (f) Peter is optimistic but not clever nor handsome.

3. **SQ** Construct and then simplify logical formulas for the following using only the \( \neg, \lor, \land \) operators.

   (a) \( p \oplus q \).
   
   (b) \( p \rightarrow \neg q \).
   
   (c) \( \neg p \leftrightarrow q \land q \).
   
   (d) \( \neg (p \rightarrow q) \).

4. **SQ** Construct and then simplify logical formulas for \( f(p, q, r) \) using only the \( \neg, \lor, \land \) operators.

   \[
   \begin{array}{ccc|c}
   p & q & r & f(p, q, r) \\
   \hline
   T & T & T & T \\
   T & T & F & F \\
   T & F & T & F \\
   T & F & F & F \\
   F & T & T & T \\
   F & T & F & F \\
   F & F & T & F \\
   F & F & F & T \\
   \end{array}
   \]

(a)
5. **SQ** Proof or disprove the equivalence of the following logical formulas.

(a) \((p \land q) \lor r\) and \(p \land (q \lor r)\).
(b) \(\neg(p \oplus q)\) and \(p \leftrightarrow q\).
(c) \(\neg(p \lor q \lor r)\) and \(\neg p \land \neg q \land \neg r\).
(d) \(p \land (p \lor q)\) and \(p \lor (p \land q)\).
(e) \((p \land q) \lor (q \land r)\) and \(q \lor (p \land r)\).

6. **SQ** Simplify the following logical formulas.

(a) \(\neg((\neg p \land (\neg q \lor p)) \lor \neg r)\).
(b) \((\neg p \land (\neg p \land r)) \lor ((q \lor (q \land r)) \land (q \lor s))\).
(c) \(((\neg p \land q) \land (q \land r)) \land \neg q\).
(d) \(\neg(\neg q \land \neg(\neg q \lor s)) \lor (q \land (r \rightarrow r))\).
(e) \(\neg p \land (p \lor q) \lor (q \lor (p \land p)) \land (p \lor \neg q)\).

7. **SQ** Express the following using only the \(\neg, \lor, \land\) operators.

(a) \(p \rightarrow q\).
(b) \(\neg(p \rightarrow (p \rightarrow q))\).
(c) (Contrapositive) \(\neg q \rightarrow \neg p\).
(d) (Converse) \(q \rightarrow p\).
(e) (Inverse) \(\neg p \rightarrow \neg q\).
(f) (Negation) \(\neg(p \rightarrow q)\).

8. **SQ** Prove or disprove the following arguments by truth tables. (Notice that the priority of operators from the highest to the lowest: 1. \(\neg\), 2. \(\land, \lor\), 3. \(\rightarrow, \leftrightarrow\))

\[
\begin{array}{|c|c|c|c|}
\hline
p & q & r & f(p, q, r) \\
\hline
T & T & T & T \\
T & T & F & F \\
T & F & T & T \\
T & F & F & T \\
F & T & T & T \\
F & T & F & F \\
F & F & T & T \\
F & F & F & F \\
\hline
\end{array}
\]

(a) \(p \rightarrow q\)
(b) \(q \rightarrow p\)
\[\therefore p \lor q\]
\[\therefore p \rightarrow q\]

(b) \(q\)
\[\therefore p\]
9. **SQ** Show that the following arguments are valid.

(a) \( \neg p \rightarrow r \land \neg s \)
\( t \rightarrow s \)
\( u \rightarrow \neg p \)
\( \neg w \)
\( u \lor w \)
\( \therefore \neg t \)
\( \neg p \lor q \rightarrow r \)
\( s \lor \neg q \)
\( \neg t \)

(b) \( p \rightarrow t \)
\( \neg p \land r \rightarrow \neg s \)
\( \therefore \neg q \)
\( p \lor q \)
\( q \rightarrow r \)

(c) \( \neg r \)
\( \neg q \rightarrow u \land s \)
\( \therefore t \)
\[ \begin{align*}
  p & \rightarrow q \\
  r \lor s & \\
  \neg s & \rightarrow \neg t \\
  \neg q & \lor s \\
  \neg s & \\
  \neg p \land r & \rightarrow u \\
  w \lor t & \\
  \therefore u & \land w
\end{align*} \]

10. T/F

(a) \[ p \land \neg p \] is an valid argument. \[ \therefore p \]

(b) \[ p \lor \neg p \] is an valid argument. \[ \therefore p \]

(c) \[ q \] \[ \therefore p \lor \neg p \]

(d) The floor is dry \[ \therefore \text{Today is sunny} \] is an valid argument.

(e) \[ (p \rightarrow q) \lor (q \rightarrow p) \] is a tautology.

(f) \[ ((p \rightarrow q) \land (q \rightarrow p)) \land \neg(p \leftrightarrow q) \] is a contradiction.

11. SQ A detective has interviewed four witnesses to a crime. From their stories, the detective has concluded that

(a) If the butler is telling the truth, then so is the cook.
(b) The cook and the gardener cannot both be telling the truth.
(c) The gardener and the handyman are not both lying.
(d) If the handyman is telling the truth then the cook is lying.

Deduce who MUST be lying? (There may be more than one liar.) Show your steps.

12. SQ Eve was killed by two person out of 4 suspects Alice, Bob, Carol and Dave. Detective Conan has the following observation

(a) If Dave didn’t kill Eve, then Alice is not a murderer implies Carol is a murderer.
(b) Bob is a murderer only if Alice killed Eve.
(c) If Dave killed Eve, then it is impossible for Bob to be a murderer.
(d) Carol is not a murderer if Bob didn’t kill Eve.

Deduce who are the murderers.

13. SQ In a remote village, there are three types of people: Knights – those who always tell the truth; Knaves – those who always lie; Spies – those who can lie or tell the truth.
(a) You have encountered three villagers, and among them there is exactly one spy, one knight and one knave. They address you as follow:

A: C is a knave.
B: A is a knight.
C: I am the spy.

What are A, B and C?

(b) You then encounter another three villagers, and among them there is exactly one spy, one knight and one knave. They address you as follow:

D: E is the spy!
E: No, F is the spy!
F: No, E is definitely the spy.

What are D, E and F?

First order logic

1. Express the following sentences using first order logic. You should define your predicates clearly and use quantifiers whenever it is possible.

   (a) Some students go to at least one tutorial of each course every week.
   (b) Not every student finishes every question in some quizzes.
   (c) At least one student go to every lecture but none of the tutorials.
   (d) Each tutor is in charge of some topics.

2. Translate the negation of each sentence in the previous question into English sentences.

3. Let $O(r)$ be ”Router r is out of services”, $B(r)$ be ”Printer r is busy”, $L(p)$ be ”Packet p is lost” and $Q(p)$ be ”Packet p is queued”. Translate the following and its negation into English.

   (a) $\exists r, (O(r) \land B(r)) \rightarrow \exists p, L(p)$.
   (b) $\forall r, B(r) \rightarrow \exists p, Q(p)$.
   (c) $\exists p, (Q(p) \land L(p)) \rightarrow \exists r, O(r)$.
   (d) $\exists r, B(r) \land \exists p, Q(p)) \rightarrow \exists p, L(p)$.

4. Express the following mathematical statements into first order logic.

   (a) Any odd prime can be written as the sum of the square of two integers.
   (b) Every positive integer can be written as the sum of the square of four integers.
   (c) Every three positive integers which satisfy $a + b = c$ must have a common factor greater than 1.
   (d) There exists a unique prime number that is even.
5. Assume the variable \( x \) has domain \( X \) and there exists an element \( a \in X \). Show that the following arguments are valid.

\[
\begin{align*}
& \forall x \in S, \neg Q(x) \to R(x) \\
\therefore & \neg P(a) \lor \neg T(a) \\
& \therefore R(a)
\end{align*}
\]

(b)

\[
\begin{align*}
& \forall x, \neg R(x) \to S(x) \\
& \neg S(a) \lor \neg U(a) \lor V(a) \\
& \forall x, V(x) \to W(x) \\
& \forall x, \neg W(x) \to U(x) \\
& \forall x, T(x) \land (\forall x, \neg R(x)) \\
& \therefore W(a)
\end{align*}
\]

6. T/F Let \( P(x, y) \) be a predicate, where \( x \) and \( y \) has non empty domain. Which of the following is true?

(a) \( \exists x, \exists y, P(x, y) \equiv \exists y, \exists x, P(x, y) \).

(b) \( \forall x, \forall y, P(x, y) \equiv \forall y, \forall x, P(x, y) \).

(c) \( \forall x, \exists y, P(x, y) \equiv \exists y, \forall x, P(x, y) \).

(d) \( \forall x, \exists y, \neg P(x, y) \equiv \neg (\exists x, \forall y, \neg P(x, y)) \).

(e) \( \forall x, \exists y, \neg P(x, y) \equiv \forall x, \neg \forall y, P(x, y) \).

(f) \( \neg \forall y, \forall x, P(x, y) \equiv \forall x, \neg \forall y, P(x, y) \).

(g) \( \forall x, \exists y, P(x, y) \to \exists y, \exists x, P(x, y) \).

(h) \( \forall x \in \mathbb{Z}^+, \exists y \in \mathbb{Z}^+, x > y \), where \( \mathbb{Z}^+ \) is the set of positive integer.

(i) There are infinitely many prime numbers can be written as:
\[
\forall p \in \mathbb{N}, \exists q \in \mathbb{N}, (\text{prime}(p) \land \text{prime}(q) \land (q > p)).
\]

Direct proof

1. Show that the product of an even number and an odd number is even.

2. Let \( a, b, k \in \mathbb{Z} \). Show that if \( a \) and \( b \) are both multiples of \( k \), then \( a + b \) is also a multiple of \( k \).

3. Let \( a, b, c \in \mathbb{N} \), where \( c \leq b \leq a \). Show that \( \binom{a}{b} \binom{b}{c} = \binom{a}{b-c} \binom{a-b+c}{c} \).

4. Show that the product of any five consecutive integers is divisible by 120. (For example, the product of 3, 4, 5, 6, and 7 is 2520, and 2520 = 120 \times 21.)

5. If \( n \) is odd, then \( n^2 - 1 \) is a multiple of 8.
Proof by contrapositive

1. Let \( p \in \mathbb{Z} \). Show that if \( p^2 \) is a multiple of 3, then \( p \) is a multiple of 3.
2. Suppose \( x \in \mathbb{R} \). If \( x^2 + 5x < 0 \) then \( x < 0 \).
3. Let \( p \in \mathbb{Z} \). Show that if \( p^k \) is even, then \( p \) is even.
4. Let \( a, b \in \mathbb{Z} \). Show that if \( a^2(b^2 - 2b) \) is odd, then \( a \) and \( b \) are odd.
5. If \( n \in \mathbb{N} \) and \( 2^n - 1 \) is prime, then \( n \) is prime.

Proof by contradiction

1. Show that \( \sqrt{2} \) is not rational.
2. Show that \( \sqrt{3} \) is not rational. (Hint: use Q1 in the previous section.)
3. Let \( a, b \in \mathbb{R} \). Show that if \( a \) is rational and \( ab \) is not rational, then \( b \) cannot be rational.
4. Show that there exist no integers \( a \) and \( b \) such that \( 18a + 6b = 1 \).
5. Suppose \( n \) students took a quiz and the average score is 80 (out of 100). Show that at least \( n/2 \) students score greater than 60.
6. Let \( a, b \in \mathbb{Z} \). Show that \( a^2 - 4b - 3 \neq 0 \).

Proof by cases

1. Show that \( |x||y| = |xy| \) for all real numbers \( x, y \).
2. Show that \( \max(x, y) + \min(x, y) = x + y \).
3. Show that \( \max(x, y) = (|x + y| + |x - y|)/2 \).
4. (Triangle inequality) Show that \( |a + b| \) is less than or equal to \( |a| + |b| \).
5. Let \( n \) is a positive integer. Show that \( n^2 - n \) is divisible by 7.

Induction

1. Prove that \( \sum_{i=1}^{n} i^2 = n(2n + 1)(n + 1)/6 \) for all \( n \geq 1 \).
2. Prove that \( \sum_{i=1}^{n} i^3 = n^2(n + 1)^2/4 \) for all \( n \geq 1 \).
3. Prove that \( \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n} \).
4. Use induction to prove that the following equation holds for all \( n \geq 2 \):

\[
(1 - \frac{1}{2})(1 - \frac{1}{3}) \cdots (1 - \frac{1}{n}) = \frac{1}{n}.
\]
5. Prove that $\sum_{k=0}^{n} \binom{k}{r} = \binom{n+1}{r+1}$, where $1 \leq r \leq n$.

6. Consider the Fibonacci sequence (i.e. $F_0 = 0, F_1 = 1, F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$). Prove that $F_1 + F_2 + F_3 + F_4 + \cdots + F_n = F_{n+2} - 1$.

7. If $n \in \mathbb{N}$ and $F_n$ is the $n^{th}$ Fibonacci number. Prove that
   \[
   \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \binom{n-3}{3} + \cdots + \binom{0}{n} = F_{n+1}.
   \]
   (For example, $\binom{6}{0} + \binom{5}{1} + \binom{4}{2} + \binom{3}{3} + \binom{2}{4} + \binom{1}{5} + \binom{0}{6} = 1 + 5 + 6 + 1 + 0 + 0 + 0 = 13 = F_{6+1}$.)

8. Prove that $n^3 + 2n$ is divisible by 3 for every positive integer $n$.

9. Use induction to prove that all element in the matrix
   \[
   \begin{pmatrix}
   1 & 2 \\
   0 & 1
   \end{pmatrix}
   \]
   are $\leq 2n$ for all positive integer $n$.

10. Let $S(n)$ be the statement: for any $n$ non negative real numbers $x_1, x_2, \ldots, x_n$,
    \[
    \frac{x_1 + x_2 + \ldots + x_n}{n} \geq \sqrt[\text{n}]{x_1x_2\ldots x_n}
    \]
    (a) Show that $S(1)$ and $S(2)$ is true.
    (b) Show that if $S(k)$ is true, $S(2k)$ is true for any positive integer $k$.
    (c) Show that if $S(k+1)$ is true, $S(k)$ is true for any positive integer $k$.

11. If you have an unlimited supply of 3 cent and 7 cent stamps,
    (a) What is the set of amounts that you can form by using just these 2 types of stamps?
    (b) What is the smallest number of 7 cent stamps so that you can still form the same set of amounts?

12. If sequence $\{a_n\}$ is defined as follows
    \[
    a_n = \begin{cases} 
    1 & \text{for } n = 0 \\
    1 + \sum_{i=0}^{n-1} a_i & \text{for } n > 0
    \end{cases}
    \]
    then use strong induction to prove that $a_n = 2^n$ for any integer $n \geq 0$.

13. Given any integer $n$ and any positive integer $d$, show that there exist integer $q$ and $r$, where $0 \leq r < d$, such that $n = dq + r$.

14. Prove that to divide up a chocolate bar with $m \times n$ squares, we need at least $mn - 1$ splits.

15. Prove by well-ordering principle that $\sum_{i=1}^{n} i = n(n + 1)/2$ for every $n \geq 0$. 

16. Use the Well-ordering principle to show that the equation \( a^2 + b^2 = 3(s^2 + t^2) \) has no non-zero integer solution.

17. Show that the equation \( 8u^4 + 4v^4 + 2w^4 = x^4 \) has no non-zero integer solution.

18. Find an ordering of all the \( n \)-bit strings in such a way that two consecutive \( n \)-bit strings differed by only one bit. This is called the **Gray code** and has many applications. The 3-ary (ternary) Gray code would use the values \{0, 1, 2\}. The \((3, n)\)-Gray code is the 3-ary Gray code with \( n \) digits. For example, the sequence of elements in the \((3, 2)\)-Gray code can be: \{00, 01, 02, 12, 10, 11, 21, 22, 20\}. Explain how to construct \((3, n)\)-Gray code.

19. Define \( F_0 = 0, F_1 = 1, F_{n+2} = F_{n+1} + F_n \) for \( n = 0, 1, 2, \ldots \) Prove that for any \( n \geq 0 \) we have \( F_n \leq \left(\frac{1 + \sqrt{5}}{2}\right)^n - 1 \).

20. Prove by induction the following is true.
\[
\sum_{i=1}^{n} ix^i = \frac{x - (n + 1)x^{n+1} + nx^{n+2}}{(1-x)^2}
\]

21. Show that a \( 6 \times n \) board \((n \geq 2)\) can be tiled with L-shaped tiles, without gap and overlapping. Each L-shaped tile covers three squares.

22. In ancient Egypt, fractions were written as sums of fractions with numerator 1. For instance, \( \frac{3}{5} = \frac{1}{2} + \frac{1}{10} \). Consider the following algorithm for writing a fraction \( \frac{m}{n} \) in this form \((1 \leq m < n)\): write the fraction \( \frac{1}{\lceil n/m \rceil} \), calculate the fraction \( \frac{m}{n} - \frac{1}{\lceil n/m \rceil} \), and if it is nonzero repeat the same step. Here \( \lceil n/m \rceil \) denotes the smallest integer not less than \( n/m \).

(a) Hackson tries to prove that the algorithm always finishes in a finite number of steps. Below is a sketch of his proof:
Observe that given a fraction \( f \) to the algorithm, value of \( f \) will be decreasing as the algorithm goes on. In other words, the new fraction \( f' \), obtained from \( f \) after applying one step of the algorithm, will be smaller than \( f \). If I prove by induction on value of \( f \), then I can assume applying the algorithm on \( f' \) will finish in finite number of steps. Hence the algorithm will finish in finite number of steps having \( f \) as input, because having \( f \) as input just require one more steps than having \( f' \) as input.

What’s wrong with the proof ?

(b) Prove the algorithm always finishes in a finite number of steps using induction.

23. You are playing a new version of the Tower of Hanoi. The rules are the same except that now you can only move disks to adjacent pole, i.e., you cannot move disks direct from pole \#1 to pole \#3 or from pole \#3 to pole \#1. For example, under the new rule, the fastest way to move 1 disk from pole \#1 to pole \#3 is 1 \( \rightarrow \) 2, 2 \( \rightarrow \) 3, requiring 2 steps.

(a) Find the fastest way to move 2 disks. State every move.

(b) Let \( R(n) \) be the minimum steps required to move \( n \) disks under the new rule. Describe your strategy of disk moving that achieves the minimum \( R \), and find a recurrence relation of \( R \). Then solve this recurrence and compute \( R(n) \). Explain your answers and show your steps.
(c) Prove by induction that your strategy actually passes through all possible configurations.

24. A sorting network of size \( n \) is a network made of \( n \) horizontal wires and some vertical comparators for sorting \( n \) numbers. When we input \( n \) distinct numbers to the left of the network in any order, the values are carried along the horizontal wires and vertical comparators to the right as outputs. Those \( n \) numbers will be output in ascending order from top to bottom. Each comparator takes 2 horizontal wires as an input and emits the smaller digit to the upper wire and larger digit to the lower wire.

![Sorting Network Example](image)

Above is an example of such network, although it does not sort correctly. First, we input 3, 2, 1 to the left of the network. Then, 3 and 2 exchange their position after the first comparator. After that, 1 and 2 exchange their position at the second comparator. Finally, nothing happens in the third comparator as 1, 3 are sorted already. Therefore the final output is 1, 3, 2.

(a) Draw sorting networks for sorting 2, 3 and 4 numbers respectively. Your network should sort all possible input sequences correctly.

(b) Prove by induction that there exists a sorting network for sorting \( n \) numbers for any \( n \geq 2 \).

(c) How many comparators are needed for sorting \( n \) numbers by the construction in (b)?

25. Bob and Alice are playing a game consisting of 5 circles and 5 squares. They take turns to select two shapes. If two identical shapes are selected, replace the two shapes with a square. At the end of the game, there is only one shape left, Alice wins if the remaining shape is a circle. Otherwise, Bob wins. Can Bob win the game?

26. Integers 1, 2, 3, 4 and 5 are written on a board. Tom picks any two of the numbers, deletes them, and writes on the board the absolute value of their difference. He repeats this procedure with the resulting 4 numbers, and so on. After he does it 4 times, only one number remains on the board. Can this number be 2?

27. This is a proof using induction. What is wrong with this proof?

Statement: For every positive integer \( n \),

\[
\sum_{i=1}^{n} i = \left( n + \frac{1}{2} \right)^2 / 2
\]

This is true when \( n = 1 \).
Assume it is true when \( n = k \).

When \( n = k + 1 \),

\[
\text{L.H.S} = \sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + k + 1
\]

\[
= \frac{1}{2} (k + \frac{1}{2})^2 + k + 1 \quad \text{By assumption}
\]

\[
= \frac{1}{2} (k^2 + k + \frac{1}{4}) + k + 1
\]

\[
= \frac{1}{2} (k^2 + k + \frac{1}{4}) + \frac{2(k+1)}{2}
\]

\[
= k^2 + k + \frac{1}{4} + 2(k + 1)
\]

\[
= \frac{(k^2 + 2k + 1) + (k + 1) + \frac{1}{4}}{2}
\]

\[
= \frac{(k + 1)^2 + 2(k+1)(\frac{1}{2}) + (\frac{1}{2})^2}{2}
\]

\[
= \frac{((k + 1) + \frac{1}{2})^2}{2} = \text{R.H.S}
\]

So the statement is proved by induction.

28. (Optional) Show that it is impossible to get from the left configuration to the right one by making sliding moves that use the empty space.

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12 \\
12 & 14 & 15 & \\
\end{array} \quad \begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12 \\
12 & 15 & 14 & \\
\end{array}
\]