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RESEARCH ARTICLE

Two Operators on the Lattice of Completely Regular Semigroup Varieties

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Abstract

In this paper, varieties of completely regular semigroups are studied. This paper is divided into six sections. Section 1 contains an introduction to varieties of completely regular semigroups and preliminaries. Most of the notation needed in this paper is given. In Section 2, the operators $\overleftarrow{(\ )}$ and $\overrightarrow{(\ )}$ on the lattice of subvarieties of varieties of completely regular semigroups are investigated. In Section 3, some further properties of the operators $\overleftarrow{(\ )}$ and $\overrightarrow{(\ )}$ are given. In Section 4, the semigroups generated by various subset of some operators are considered. In Section 5, the operators $\overleftarrow{(\ )}$ and $\overrightarrow{(\ )}$ are used in finding the join of two given varieties. The word problem for free objects in the variety $OLBG$ is considered in Section 6 using the operator $\overleftarrow{(\ )}$.

1. Introduction and preliminaries

Completely regular semigroups (unions of groups) may be regarded as algebras with the operations of (binary) multiplication and (unary) inversion. As such they form a variety CR defined by the indentities

$$(ab)c = a(bc), a = aa^{-1}a, aa^{-1} = a^{-1}a, (a^{-1})^{-1} = a.$$

Varieties are usually presented by identities or by the structure of their algebras. Libor Polák [see 11, 12, 13] used another approach by presenting varieties by means of a "good" description of the corresponding fully invariant congruences on the absolutely free algebra. In fact, he defined some operators by constructing new fully invariant congruences from the given ones. These operators proved to be very effective in studying the lattice $\mathcal{L}(CR)$. In this paper, we will study two operators $\overleftarrow{(\ )}$ and $\overrightarrow{(\ )}$ which were only defined but not studied further in [11]. These two operators play an important role in the whole paper.

In [4], P. R. Jones studied the Mal′cev products of varieties of completely regular semigroups. In [16], S. H. Zhang gave the relations between some

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operators and Mal$'$cev products on varieties of completely regular semigroups. In our paper, we find the relation between the operator $\overleftarrow{(\ )}$ and the Mal$'$cev product on varieties of completely regular semigroups.

The operators $\overline{(\ )}, (\ )^+, (\ )^0, (\ )^1, C, L$ and some others on the lattice $\mathcal{L}(CR)$ of varieties of completely regular semigroups have played an important role in recent studies of the lattice $\mathcal{L}(CR)$. The nature of the semigroups generated by various subset of the set of these operators will be usefull in the studies of the lattice $\mathcal{L}(CR)$. In [9], M. Petrich and N. R. Reilly determined the semigroups generated by various subset of $\{\overline{(\ )}, (\ )^+, (\ )^0, (\ )^1, C, L\}$. It is natural to consider the semigroups generated by some subsets of the set of these operators, the operator $\overleftarrow{(\ )}$ and the operator $\overrightarrow{(\ )}$, and we get some results.

Given two varieties V_1 and V_2 of completely regular semigroups, how to characterize the join of V_1 and V_2 in $\mathcal{L}(CR)$ is a basic problem in the local studies of the lattice $\mathcal{L}(CR)$. In [2], [3] and [4], the reader can find some materials about the join of two varieties of completely regular semigroups. In our paper, we deal this problem by using the operators $\overleftarrow{(\ )}$ and $\overrightarrow{(\ )}$ on the lattice $\mathcal{L}(CR)$.

In [7], F. Pastijn proved that the lattice $\mathcal{L}(CR)$ is an arguesian lattice, and hence it is a modular lattice. Since the lattice $\mathcal{L}(G)$ of the subvarieties of the variety of groups is not a distributive lattice, the equality

$$W \cap (V_1 \vee V_2) = (W \cap V_1) \vee W \cap V_2)$$

doesn't hold in general. It is interesting to find the conditions under which the above equality holds. We obtain some sufficient conditions.

As an application, we also consider the word problem for free left cryptic orthogroups.

In a completely regular semigroup S, we use the following notation. If $x \in S$, then x^{-1} is the inverse of x in the maximal subgroup of S containing x. In addition, let $x^0 = xx^{-1}$. Also, $E(S)$ denotes the set of idempotents of S and $Con\,S$ denotes the lattice of congruences on S.

Certain congruences on a completely regular semigroup are particularly important. Let $\rho \in Con\,S, S \in CR$. Then ρ is said to be idempotent separating if $e, f \in E(S)$ and $e \rho f$ imply $e = f$, while ρ is idempotent pure if $e \in E(S)$ and $e \rho a$ imply $a \in E(S)$. We will denote by $\mu = \mu_S$ (respectively $\tau = \tau_S$) the maximum idempotent separating (respectively idempotent pure) congruence on S. Also, for any equivalence relation λ on S, we denote by λ^0 the largest congruence on S contained in λ. It is sometimes useful to remember that $\mu = \mathcal{H}^0$.

The term variety means variety of completely regular semigroups as algebras with multiplication and inversion. We use the following notation for various varieties:

T:	one element semigroups.
LZ:	left zero semigroups.
RZ:	right zero semigroups.
ReB:	rectangular bands.
SL:	semilattices.
LRB:	left regular bands.
RRB:	right regular bands.
RB:	regular bands.
B:	bands.
G:	groups.
CS:	completely simple semigroups.
O:	orthogroups (orthodox completely regular semigroups).
LRO:	orthogroups of which the band of the idempotents is left regular band.
RRO:	orthogroups of which the band of the idempotents is right regular band.
BG:	completely regular semigroups on which Green's relation $\mathcal{H}$ is a congruence.
OBG:	orthogroups on which Green's relation $\mathcal{H}$ is a congruence.
LBG:	completely regular semigroups on which Green's relation $\mathcal{H}$ is a left congruence.
RBG:	completely regular semigroups on which Green's relation $\mathcal{H}$ is a right congruence.
$OLBG$:	orthogroups on which Green's relation $\mathcal{H}$ is a left congruence.
$ORBG$:	orthogroups on which Green's relation $\mathcal{H}$ is a right congruence.
CR:	completely regular semigroups.
Moreover,	
$\mathcal{L}(CR)$:	the lattice of subvarieties of CR,
FV:	the (relatively) free completely regular semigroup on a countably infinite set in a variety V.
F_n:	the free completely regular semigroup on a set of n elements.

Recall that a congruence $\sim$ of a universal algebra (A, F) is called fully invariant (briefly f.i.) if for any $a, b \in A$ and any endomorphism α of (A, F) it is the case that $a \sim b$ implies $\alpha(a) \sim \alpha(b)$. It is well-known that there is a one-to-one correspondence between f. i. congruences on the absolutely free algebra $F_\tau(\omega)$ of a given type τ on $\{x_1, x_2, \ldots\}$ and varieties of this type. It is given by the map $\sim \longrightarrow HSP(F_\tau(\omega)/\sim)$.

A solution of the word problem for free objects of a given variety V of type τ consists of a description of an algorithm which decides for given words $s, t \in F_\tau(\omega)$ whether the identity $s = t$ holds in all algebras of V or not. This is the same as giving an effective description of the f. i. conruence on $F_\tau(\omega)$ corresponding to V. In what follows, we consider only varieties of completely regular semigroup. The role of $F_\tau(\omega)$ will be played by the free unary semigroup U.

Let $X = \{x_1, x_2, \ldots\}$ be an infinite countable set of variables, and $P = X \cup \{(,)^{-1}\}$ where "(" and "$)^{-1}$" are two distinct elements not in X. By U denote the least subset of F, the free semigroup on P, satisfying

(1) $x_1, x_2, \ldots \in U$,
(2) $a, b \in U$ implies $a.b \in U$ and,
(3) $a \in U$ implies $(a)^{-1} \in U$.

Elements of U are called words. Let a^0 stand for $a.(a)^{-1}, a^{-n}$ for $(a^{-1})^n$ (n a positive integer), and let $a(p_1, p_2, \ldots)$ denote the word obtained from a by substituting p_i for any occurrence of x_i in $a (i = 1, 2, \ldots)$.

Let $f \in F$ be a segment of $a \in U$ (i.e. $a = efg$ for some $e, g \in F$). Brackets in a are matched in pairs. We write $\hat{f}$ for the word of U which we obtain by deleting all nonmatched "(" and "$)^{-1}$" in f.

For a word $a \in U$, we define the following invariants:

$E(a)$: the set of all variables of a;
$C(a)$: the cardinality of the set $E(a)$;
$r(a)$: the group reduced word of a. This may be obtained (as is well-known) by applying $(xy)^{-1} = y^{-1}x^{-1}$ to write words as products of generators and inverses of generators and then removing all occurrences of xx^{-1} and $x^{-1}x$ for any $x \in X$;
$\langle a\rangle_L$: the sequence of all variables of a ordered as their first occurrences from the left in a;
$\langle a\rangle_R$: the sequence of all variables of a ordered as their first occurrences from the right in a;
$0(a) = \hat{b}$ where b is the longest initial segment of a in $C(a)-1$ variables;
$1(a) = \hat{c}$ where c is the longest final segment of a in $C(a)-1$ variables.

Put $0^{k+1}(a) = 0(0^k(a)), 1^{k+1}(a) = 1(1^k(a))$ (k a positive integer). Put $U_n = \{a \in U\colon\ E(a) \subseteq \{x_1, \ldots x_n\}\}$, $U_n^* = \{a \in U\colon\ C(a) \leq n\}$.

Let Σ be a set of identities (i.e. pairs of U). The corresponding variety of completely regular semigroups will be denoted by $V(\Sigma)$ or $[\Sigma]$ (i.e. $V(\Sigma)$ is determined by Σ and $(xy)z = x(yz), xx^{-1}x = x, xx^{-1} = x^{-1}x, (x^{-1})^{-1} = x$). If V is a variety of completely regular semigroups, then the corresponding f. i. congruence on U will be denoted by $\sim_V$. We write $\sim(\Sigma)$ instead of $\sim_{V(\Sigma)}$. Clearly $V \subseteq W$ iff $\sim_V \supseteq \sim_W$.

The lattice of all f. i. congruence on U will be denoted by $FICU$.

It is well known that

$$[\sim_{CR}, U \times U] \to \mathcal{L}(CR), \sim \to V(\sim)$$

and

$$\mathcal{L}(CR) \to [\sim_{CR}, U \times U], V \to \sim_V$$

are mutually inverse antiisomorphisms. (Where the interval $[\sim_{CR}, U \times U]$ denotes the lattice of fully invariant congruences on U which contain $\sim_{CR}$).

Note that $\sim_{SL} = E = \{(a, b) \in U \times U : E(a) = E(b)\}$ and $\sim_{LZ} = h = \{(a, b) \in U \times U : h(a) = h(b)\}$.

The operators $\overleftarrow{(\ )}$ and $\overrightarrow{(\ )}$ are discussed in Section 2 and Section 3. The semigroup generated by the set of some operators is described in Section 4. Section 5 contains the material about the join of two varieties. The word problems for free leftcryptic orthogroup is studied in Section 6.

The following Lemma will be used frequently, and be used without further notice.

Lemma A. *Let $S \in CR$. Then $\tau \cap \mathcal{H} = 1_S$.*

Proof. Let $a, b \in H_e, e \in E(S)$ and $(a, b) \in \tau$. Then $(aa^{-1}, ba^{-1}) \in \tau$. Since τ is idempotent pure, $ba^{-1} \in E(S)$. Since H_e is a group, it follows that $ba^{-1} \in H_e$ and hence $ba^{-1} = e$. Therefore $a = b$, as required. ■

Let $\sim \in U \times U$. The following definitions of relations $\sim_0, \sim_1, \overline{(\ )}, \overleftarrow{\sim}$ and $\overrightarrow{\sim}$ on U can be found in [11]:

$a \sim_0 b$ iff there are $c, d \in U$ such that $c \sim d, a = 0(c), b = 0(d)$;
$a \sim_1 b$ iff there are $c, d \in U$ such that $c \sim d, a = 1(c), b = 1(d)$;
$a \overline{\sim} b$ iff $E(a) = E(b), a \sim b, 0(a) \overline{\sim} 0(b), 1(a) \overline{\sim} 1(b)$;
$a \overleftarrow{\sim} b$ iff $E(a) = E(b), a \sim b, 0(a) \overleftarrow{\sim} 0(b)$;
$a \overrightarrow{\sim} b$ iff $E(a) = E(b), a \sim b, 1(a) \overrightarrow{\sim} 1(b)$;

The last three definitions are inductive ones with respect to $C(a)$ and the conditions concerning $0(a), 0(b), 1(a)$ and $1(b)$ drop off in the case $C(a) = 1$.

Obviously $\overline{\sim}, \overleftarrow{\sim}, \overrightarrow{\sim} \subseteq E$.

Let $V = V(\sim)$. Put $\overline{V} = V(\overline{\sim}), \overleftarrow{V} = V(\overleftarrow{\sim}), \overrightarrow{V} = V(\overrightarrow{\sim}), V_0 = V(\sim_0), V_1 = V(\sim_1)$, and $V \rho W$ iff $\overline{V} = \overline{W}, V \rho_l W$ iff $\overleftarrow{V} = \overleftarrow{W}$, and $V \rho_r W$ iff $\overrightarrow{V} = \overrightarrow{W}$.

2. The properties of the operators $\overleftarrow{\sim}$ and $\overrightarrow{\sim}$

The main result of this section is an analogue of the Polák's result in [11]. The operator $\overline{(\ )}$ is dual with itself, but the operator $\overleftarrow{\sim}$ is dual with $\overrightarrow{\sim}$.

We will use the following lemmas stated in [11].

Lemma 2.1 [11, Lemma 1]**.** (1) *If $v \in F$ is an initial segment of $u \in U$, then $\hat{v}w \sim_{CR} u$ for some $w \in U$.*

(2) *If $u, v \in U, E(v) \subseteq E(u)$, then $(uv)^0 u \sim_{CR} u$ and there exists $w \in U$ such that $uvw \sim_{CR} u$.*

(3) *Let $u, v \in U, u \sim_{CR} v, x \in E(u) \backslash E(0(u)), y \in E(v) \backslash E(0(v))$. Then $E(u) = E(v), 0(u) \sim_{CR} 0(v)$, and $x = y$.*

(4) *Let $u \in U, x \in E(u) \backslash E(0(u)), y \in E(u) \backslash E(1(u))$. Then there is $u' \in U$ such that $E(u') \subseteq E(u)$ and $u \sim_{CR} 0(u) x u' y 1(u)$.*

(5) *If $u, v \in U, E(u) = E(v), 0(u) \sim_{CR} 0(v), 1(u) \sim_{CR} 1(v)$, then $u^0 \sim_{CR} v^0$.*

Lemma 2.2 [11, Lemma 2]**.** *A relation $\sim$ on U is a f. i. congruence if and only if*

(1) $\sim$ *is an equivalence relation,*
(2) *for any* $a, b, c \in U, a \sim b$ *implies* $ca \sim cb$ *and* $ac \sim bc$,
(3) *for any* $a, b \in U, a \sim b$ *implies* $(a)^{-1} \sim (b)^{-1}$, *and*
(4) *for any* $a, b, p_1, p_1, \ldots \in U, a \sim b$ *implies* $a(p_1, p_2, \ldots) \sim b(p_1, p_2 \ldots)$.

Remark 2.2′. It is a well-known fact that any congruence on completely regular semigroup respects $()^{-1}$.Thus for a relation $\sim \supseteq \sim_{CR}$, the condition (3) of Lemma 2.2 can be omitted.

Lemma 2.3 [11, Lemma 3]**.** (1) *For any* $a, b \in U$

$$0(ab) = \begin{cases} 0(a), & \text{if } E(a) = E(ab) \\ a.0^{n-k}(b), & \text{if } E(a) \neq E(ab) \end{cases}$$

where $n = c(b)$ *and* k *is determined by* $\langle b \rangle_L = (x_{i_1}, \ldots, x_{i_n}), E(a) \cup \{x_{i_1}, \ldots, x_{i_k}\} \subset E(a) \cup \{x_{i_1}, \ldots x_{i_{k+1}}\} = E(ab)$.

(2) *For any* $a, p_1, p_2, \ldots \in U$,

$$0(a(p_1, p_2, \ldots)) = (0^{n-k}(a))(p_1, p_2, \ldots).0^{m-l}(p_{i_{k+1}}),$$

where $n = C(a)$ *and* k, m, l *are determined by:* $\langle a \rangle_L = (x_{i_1}, \ldots, x_{i_n}), E(p_{i_1}) \cup \cdots \cup E(p_{i_k}) \subset E(p_{i_1}) \cup \cdots \cup E(p_{i_{k+1}}) = E(a(p_1, p_2, \ldots)), m = C(p_{i_{k+1}}), \langle p_{i_{k+1}} \rangle_L = (x_{j_1}, \ldots x_{j_m}), E(p_{i_1}) \cup \cdots \cup E(p_{i_k}) \cup \{x_{j_1}, \ldots x_{j_l}\} \subset E(p_{i_1}) \cup \cdots \cup E(p_{i_k}) \cup \{x_{j_1}, \ldots x_{j_{l+1}}\} = E(a(p_1, p_2, \ldots))$.

Now we begin to discuss the main result of this section.

Theorem 2.4. (1) *If* $\sim \in FICU$, *then* $\overleftarrow{\sim} \in FICU$.

(2) *For any f. i. congruences* $\sim$ *and* $\approx$ *on* U

(i) $\overleftarrow{\sim} \subseteq \sim$,

(ii) $\overleftarrow{\overleftarrow{\sim}} = \overleftarrow{\sim}$

(iii) $\sim \subseteq \approx$ *implies* $\overleftarrow{\sim} \subseteq \overleftarrow{\approx}$,

(iv) $\overleftarrow{\sim} = \overleftarrow{\sim \cap E \cap h}$.

(3) *For any non-void family* $(\sim_i)_{i \in I}$ *of f. i. congruences on* U

(i) $\bigcap \overleftarrow{\sim_i} = \overleftarrow{\cap \sim_i}$,

(ii) $\bigvee \overleftarrow{\sim_i} \subseteq \overleftarrow{\vee \sim_i}$, *and*

(iii) *if* $\sim_i \supseteq \sim_{CR}$ *for any* $i \in I$, *then* $\bigvee \overleftarrow{\sim_i} \supseteq \overleftarrow{\vee \sim_i}$.

(4) *The relation* ρ_l *is a complete lattice congruence on the lattice* $\mathcal{L}(CR)$.

Proof. (1). We shall verify successively the conditions of Lemma 2.2. To prove (i) note that $\overset{\leftarrow}{\sim}|U_1^* = (\sim |U_1^*) \cap E$ is an equivalence relation on U_1^*, and if $\overset{\leftarrow}{\sim}|U_n^*$ is an equivalence relation on U_n^*, then $\overset{\leftarrow}{\sim}|U_{n+1}^*$ is such a relation on U_{n+1}^*.

To prove (ii) we must show that $a\overset{\leftarrow}{\sim}b$ implies $ca\overset{\leftarrow}{\sim}cb$ and $ac\overset{\leftarrow}{\sim}bc$.

We first show that $a\overset{\leftarrow}{\sim}b$ implies $ac\overset{\leftarrow}{\sim}bc$ by using induction with respect to $C(ac)$. To prove this, note that $a\overset{\leftarrow}{\sim}b$ gives $E(a) = E(b), a \sim b, 0(a)\overset{\leftarrow}{\sim}0(b)$.This yields $E(ac) = E(bc), ac \sim bc$. It remains to prove that

$$a\overset{\leftarrow}{\sim}b \textit{ implies } 0(ac)\overset{\leftarrow}{\sim}0(bc).$$

If $C(ac) = 1$, then there is nothing to prove.

Suppose that $a\overset{\leftarrow}{\sim}b$ implies $0(ac)\overset{\leftarrow}{\sim}0(bc)$ for $C(ac) \leq n$ and let $C(ac) = n+1$. By Lemma 2.3(1), we have

$$0(ac) = \begin{cases} 0(a), & \text{if } E(a) = E(ac) \\ a.0^k(c), & \text{otherwise} \end{cases}$$

$$0(bc) = \begin{cases} 0(b), & \text{if } E(b) = E(bc) \\ b.0^l(c), & \text{otherwise} \end{cases}$$

The numbers k, l are given in Lemma 2.3(1). If $C(a) = n+1$,then $0(ac) = 0(a), 0(bc) = 0(b)$. Thus $0(a)\overset{\leftarrow}{\sim}0(b)$ yields $0(ac)\overset{\leftarrow}{\sim}0(bc)$. In the opposite case, we have $k = l$ since $E(a) = E(b)$. Thus the inductive asssumption yields $0(ac) = a.0^k(c)\overset{\leftarrow}{\sim}b.0^k(c) = 0(bc)$, and therefore $ac\overset{\leftarrow}{\sim}bc$.

Then we consider that $a\overset{\leftarrow}{\sim}b$ implies $ca\overset{\leftarrow}{\sim}cb$.Suppose that $a\overset{\leftarrow}{\sim}b$. Then obviously $E(ca) = E(cb), ca \sim cb$. It remains to prove that

$$a\overset{\leftarrow}{\sim}b \textit{ implies } 0(ca)\overset{\leftarrow}{\sim}0(cb) \tag{$*$}$$

We use induction with respcet to $C(ca)$. If $C(ca) = 1$, then there is nothing to prove.

Suppose that the implication (*) holds for $C(ca) \leq n$ and let $C(ca) = n+1$. By Lemma 2.3(1), we have:

$$0(ca) = \begin{cases} 0(c), & \text{if } E(c) = E(ca) \\ c.0^k(a), & \text{otherwise} \end{cases}$$

$$0(cb) = \begin{cases} 0(c), & \text{if } E(c) = E(cb) \\ c.0^l(b), & \text{otherwise} \end{cases}$$

The numbers k, l are given in Lemma 2.3(1). If $C(c) = n+1$, then $0(ca) = 0(cb) = 0(c)$. In the opposite case, we have: $E(0(a)) = E(0(b)), E(0^2(a)) = (0^2(b))$, ... since $0(a)\overset{\leftarrow}{\sim}0(b), 0^2(a)\overset{\leftarrow}{\sim}0^2(b)$, ... This gives $\langle a\rangle_L = \langle b\rangle_L$, and therefore $k = l$. Thus the inductive assumption yields $0(ca) = c.0^k(a) \overset{\leftharpoonup}{\sim} c.0^k(b) = 0(cb)$.

Concerning (iii) we note that for any $c \in U$ it is the case that $0((c)^{-1}) = 0(c), E((c)^{-1}) = E(c)$, and therefore $a \overset{\leftharpoonup}{\sim} b$ gives $(a)^{-1} \overset{\leftharpoonup}{\sim} (b)^{-1}$.

Concerning (iv) we note firstly that $a \overset{\leftharpoonup}{\sim} b$ gives $E(a(p_1, p_2, \ldots)) = E(b(p_1, p_2, \ldots))$ and $a(p_1, p_2, \ldots) \sim b(p_1, p_2, \ldots)$. We have to prove that $a \overset{\leftharpoonup}{\sim} b$ implies $0(a(p_1, p_2, \ldots)) \overset{\leftharpoonup}{\sim} 0(b(p_1, p_2, \ldots))$. By Lemma 2.3(2), we have:

$$0(a(p_1, p_2, \ldots)) = (0^{n-k}(a))(p_1, p_2, \ldots)0^{m-l}(p_{i_{k+1}}).$$

The numbers n, k, m, l are determined in the mentioned Lemma. Since $\langle a \rangle_L = \langle b \rangle_L$, we have:

$$0(b(p_1, p_2, \ldots)) = (0^{n-k}(b))(p_1, p_2, \ldots)0^{m-l}(p_{i_{k+1}}).$$

Using induction with respect to $C(a)$ and the statement (ii), one gets easily the required results.

In the proofs of (2) and (3) of Theorem 2.4, we use induction with respect to the number of variables in the words considered. Clearly

$$\sim = \bigcup_{n=1}^{\infty} \sim |U_n^*.$$

(2). (i) and (ii).

The statement (i) and inclusion "$\subseteq$" in (ii) follow from the definition of the operator $\overleftarrow{(\ )}$.

Clearly $\overleftarrow{\overset{\leftharpoonup}{\sim}} |U_1^* = \overset{\leftharpoonup}{\sim} |U_1^*$. Suppose that $\overset{\leftharpoonup}{\sim} |U_n^* \subseteq \overleftarrow{\overset{\leftharpoonup}{\sim}} |U_n^*$. Let $a \overset{\leftharpoonup}{\sim} b, C(a) = n + 1$. Then $E(a) = E(b)$ and by the inductive assumption, $(0(a), 0(b)) \in \overset{\leftharpoonup}{\sim} |U_n^* \subseteq \overleftarrow{\overset{\leftharpoonup}{\sim}} |U_n^*$. Thus we have $a \overleftarrow{\overset{\leftharpoonup}{\sim}} b$.

(3) (i). Clearly $\overleftarrow{\cap \sim_i} |U_1^* = (\bigcap \sim_i |U_1^*) \bigcap E = \bigcap \overset{\leftharpoonup}{\sim}_i |U_1^*$. Suppose that $\overleftarrow{\cap \sim_i} |U_n^* = \bigcap \overset{\leftharpoonup}{\sim}_i |U_n^*$, and let $C(a) = n + 1$. Then: $(a, b) \in \overleftarrow{\cap \sim_i} \Leftrightarrow E(a) = E(b), (a, b) \in \cap \sim_i, (0(a), 0(b)) \in \overleftarrow{\cap \sim_i} |U_n^* = \bigcap \overset{\leftharpoonup}{\sim}_i |U_n^*$ (by the induction assumptive) $\Leftrightarrow$ for any $i \in I, E(a) = E(b), (a, b) \in \sim_i, (0(a), 0(b)) \in \overset{\leftharpoonup}{\sim}_i \Leftrightarrow (a, b) \in \bigcap \overset{\leftharpoonup}{\sim}_i$.

(2) (iii). It follows immediately from (3)(i).

(2) (iv). By (2) (iii), the operator $\overleftarrow{(\ )}$ is an isotone one. It remains to prove the inclusion $\overset{\leftharpoonup}{\sim} \subseteq \overleftarrow{\sim \cap E \cap h}$. Clearly, it holds on U_1^*. Suppose that the inclusion holds on U_n^*. Let $(a, b) \in \overset{\leftharpoonup}{\sim}, C(a) = n + 1$. Then we have $E(a) = E(b), a \sim b, (0(a), 0(b)) \in \overset{\leftharpoonup}{\sim} \subseteq \overleftarrow{\sim \cap E \cap h}$ (by the inductive assumption). Further, $(0^2(a), 0^2(b)), \ldots, (0^n(a), 0^n(b)) \in \overset{\leftharpoonup}{\sim} \subseteq E$. Now $h(a)$ and $h(b)$ are the only elements of $E(0^n(a))$ and $E(0^n(b))$ respectively, and so

$(a,b) \in h$. Combining this with the above considerations, we have $(a,b) \in \overleftarrow{\sim \cap E \cap h}$.

(3) (ii). By (2) (iii), the operator $\overleftarrow{(\;)}$ is an isotone one, and so $\overleftarrow{\sim_i} \subseteq \overleftarrow{\vee \sim_i}$. Therefore $\bigvee \overleftarrow{\sim_i} \subseteq \overleftarrow{\vee \sim_i}$.

(3) (iii). Let $(a,b) \in \overleftarrow{\vee \sim_i}$. Then $E(a) = E(b), (a,b) \in \vee \sim_i, (0(a), 0(b)) \in \overleftarrow{\vee \sim_i}$. There are $i_1, \ldots, i_m \in I, c_1, \ldots, c_{m+1} \in U$ such that

$$a = c_1 \sim_{i_1} c_2 \sim_{i_2} \cdots \sim_{i_m} c_{m+1} = b. \tag{$*$}$$

Let $x \in E(a) = E(b)$. In (*) substitute the variable x for all variables different from those of $E(a) = E(b)$. We get

$$a = c_1' \sim_{i_1} c_2' \sim_{i_2} \cdots \sim_{i_m} c_{m+1}' = b \tag{$**$}$$

where $E(c_1'), \ldots, E(c_{m+1}') \subseteq E(a)$.

Suppose that $C(a) = 1$. By(**), we get $(a,b) \in \vee \overleftarrow{\sim_i}$. Suppose that $\overleftarrow{\vee \sim_i} | U_n^* \subseteq \bigvee \overleftarrow{\sim_i} | U_n^*, C(a) = n+1$. By Lemma 2.1(4), $a \sim_{CR} 0(a)xa'y1(a)$, $b \sim_{CR} 0(b)xb'y'1(b)$ for some $a', b' \in U, E(a'), E(b') \subseteq E(a) (x \in E(a) \backslash E(0(a)) = E(b) \backslash E(0(b)), y \in E(a) \backslash E(1(a)), y' \in E(b) \backslash E(1(b))$.

Multiplying (**) by $(0(a)x)^0$ on the left, we get $((0(a)x)^0 a, (0(a)x)^0 b) \in \bigvee \overleftarrow{\sim_i}$. By the inductive assumption, $(0(a), 0(b)) \in \bigvee \overleftarrow{\sim_i}$, and hence

$$((0(a)x)^0 b, (0(b)x)^0 b) \in \vee \overleftarrow{\sim_i}.$$

Using the above decompositions of a and b, we get $b \sim_{CR} (0(b)x)^0 b \bigvee \overleftarrow{\sim_i} (0(a)x)^0 b \bigvee \overleftarrow{\sim_i} (0(a)x)^0 a \sim_{CR} a$. Thus $(a,b) \in \bigvee \overleftarrow{\sim_i}$.

(4) This is immmediate from (3). ■

Remark 2.4′. The dual of Theorem 2.4 also holds.

By the well-known fact that there is a one-to-one correspondence between f. i. congruences on the absolutely free algebra $F_\tau(\omega)$ of a given type τ on $\{x_1, x_2, \ldots\}$ and varieties of this type, we have the following.

Corollary 2.5. *Let* $V_i \in \mathcal{L}(CR)$ *for* $i \in I, I \neq \emptyset$. *Then*

(i) $\overleftarrow{\vee V_i} = \bigvee \overleftarrow{V_i}$, *(i)′* $\overrightarrow{\vee V_i} = \bigvee \overrightarrow{V_i}$,

(ii) $\overleftarrow{\cap V_i} = \bigcap \overleftarrow{V_i}$, *(ii)′* $\overrightarrow{\cap V_i} = \bigcap \overrightarrow{V_i}$.

3. Further properties of the operators $\overleftarrow{\sim}$ and $\overrightarrow{\sim}$

It may be seen that a fully invariant congruence $\sim$ on U satisfies $\sim_0 \subseteq E$ if and only if $\sim \subseteq \sim_{LRB}$ and, similarly, $\sim_1 \subseteq E$ if and only if $\sim \subseteq \sim_{RRB}$. The argument is that the word problem for $U/\sim_{LRB}$ and $U/\sim_{RRB}$ solved by:

$$u \sim_{LRB} v \text{ iff } E(0^k(u)) = E(0^k(v)) \text{ for } 0 \le k \le C(u),$$
$$u \sim_{RRB} v \text{ iff } E(1^k(u)) = E(1^k(v)) \text{ for } 0 \le k \le C(u).$$

For details see [12, Theorem 1.2(5)].

Lemma 3.1 [5, Lemma 1.5]. *Let $\sim \in FICU$ be such that $\sim_{CR} \subseteq \sim \subseteq \sim_{LRB}$. For $u, v \in U$, the following holds:*

$$u \sim \mathcal{R} v \sim \text{ iff } E(u) = E(v) \text{ and } 0(u) \sim_0 0(v) \text{ in case } C(u) \ge 2.$$

Lemma 3.1′ [5, Lemma 1.5]. *Let $\sim \in FICU$ be such that $\sim_{CR} \subseteq \sim \subseteq \sim_{RRB}$. For $u, v \in U$, the following holds:*

$$u \sim \mathcal{L} v \sim \text{ iff } E(u) = E(v) \text{ and } 1(u) \sim_1 1(v) \text{ in case } C(u) \ge 2.$$

Note that for any $\sim \in FICU, \overleftarrow{\sim} \subseteq \sim_{LRB}$ ($\overrightarrow{\sim} \subseteq \sim_{RRB}$).

Theorem 3.2. *If $\sim, \approx \in FICU, \sim_{CR} \subseteq \sim, \approx$, then the following conditions are equivalent.*

(i) *for each $a, b \in U, a^0 \sim a$ implies $a^0 \approx a$, and $a^0 \sim a^0b^0, b^0 \sim b^0a^0, E(a) = E(b)$ implies $a^0 \approx a^0b^0, b^0 \approx b^0a^0$.*

(ii) $\overleftarrow{\sim} \subseteq \overleftarrow{\approx}$.

Proof. To establish (ii) $\Rightarrow$ (i), it is enough to observe that $\{a \in U\colon a^0 \sim a\} = \{a \in U\colon a^0 \overleftarrow{\sim} a\}$ and $\{(a,b) \in U \times U\colon E(a) = E(b), a^0 \sim a^0b^0, b^0 \sim b^0a^0\} = \{(a,b) \in U \times U\colon E(a) = E(b), a^0 \overleftarrow{\sim} a^0b^0, b^0 \overleftarrow{\sim} b^0a^0\}$ which are clear from the definition $\overleftarrow{\sim}$.

To prove (i)$\Rightarrow$(ii), assume that $\sim, \approx \in FICU, \sim_{CR} \subseteq \sim, \approx, \{a \in U\colon a^0 \sim a\} \subseteq \{a \in U\colon a^0 \approx a\}$ and $\{(a,b) \in U \times U\colon E(a) = E(b), a^0 \sim a^0b^0, b^0 \sim b^0a^0\} \subseteq \{(a,b) \in U \times U\colon E(a) = E(b), a^0 \approx a^0b^0, b^0 \approx b^0a^0\}$. Using induction with respect to $C(a)$, we show that $a \overleftarrow{\sim} b$ implies $a \overleftarrow{\approx} b$.

If $C(a) = 1, a \overleftarrow{\sim} b$, then $E(a) = E(b), a \sim b$. Now $ab^{-1} \sim b^0$ and $(ab^{-1})^0 \sim ab^{-1}$. This gives $(ab^{-1})^0 \approx ab^{-1}$. Since $E(a) = E(b) = E(ab^{-1})$ and $C(a) = 1, a^0 \sim_{CR} b^0 \sim_{CR} (ab^{-1})^0$ by Lemma 2.1(5). This gives $(ab^{-1})^0 b \sim_{CR} b^0b \sim_{CR} b$. Thus

$$a \sim_{CR} aa^0 \sim_{CR} ab^0 \sim_{CR} ab^{-1}b \approx (ab^{-1})^0 b \sim_{CR} b,$$

so $a \approx b$ and $a \overleftarrow{\approx} b$.

Suppose now that $C(a) = n \ge 2, a \overleftarrow{\sim} b$. Then $E(a) = E(b), 0(a) \overleftarrow{\sim} 0(b), a \sim b$. By the inductive hypothesis, $0(a) \overleftarrow{\approx} 0(b)$, then by Lemma 3.1, $a^0 \overleftarrow{\approx} \mathcal{R} b^0 \overleftarrow{\approx} \mathcal{R} (ab^{-1})^0 \overleftarrow{\approx}$ (since $\overleftarrow{\approx} \subseteq (\overleftarrow{\approx})_0$ and $E(a) = E(b)$). Now $ab^{-1} \sim b^0$ gives $(ab^{-1})^0 \sim ab^{-1}$, and so

$$(ab^{-1})^0 b \approx ab^{-1}b \approx ab^0. \tag{1}$$

Since $b^0 \overleftarrow{\approx} R(ab^{-1})^0 \overleftarrow{\approx}$, it follows that

$$(ab^{-1})^0 b \sim_{CR} (ab^{-1})^0 b^0 b \overleftarrow{\approx} b^0 b \sim_{CR} b. \tag{2}$$

Now we show that $ab^0 \approx a$. $a \sim b$ gives $a^0 \sim b^0$ and $a^0 \sim a^0 b^0, b^0 \sim b^0 a^0$, and hence $a^0 \approx a^0 b^0, b^0 \approx b^0 a^0$ since $E(a) = E(b)$. Thus

$$ab^0 \sim_{CR} aa^0 b^0 \approx aa^0 \sim_{CR} a. \tag{3}$$

So we have

$$\begin{aligned} a &\approx ab^0 && by(3) \\ &\approx (ab^{-1})^0 b && by(1) \\ &\approx b && by(2) \end{aligned}$$

Therefore $a \overleftarrow{\approx} b$. ■

Lemma 3.3. *Let* $\sim, \approx \in FICU, \sim_{CR} \subseteq \sim \subseteq \approx$. *Then* $\{a \in U\colon\ a \sim a^0\} = \{a \in U\colon\ a \approx a^0\}$ *if and only if* $\approx / \sim$ *is idempotent pure on* $U/\sim$.

Proof. "⇐". We only need to show that $a \approx a^0$ implies $a \sim a^0$. Suppose $a \approx a^0$, then $(a \sim, a^0 \sim) \in \approx / \sim$. Since $\approx / \sim$ is idempotent pure, it follows that $a \sim$ is an idempotent, and hence $a \sim a^0$.

"⇒". Let $(a \sim, b \sim) \in \approx / \sim$ be such that $b \sim$ is an idempotent. Then $a \approx b, b \sim b^0$. So $a \approx b^0$ which yields $a \approx a^0$. Thus $a \sim a^0$ and $\approx / \sim$ is idempotent pure. ■

Lemma 3.4. *Let* $\sim, \approx \in FICU, \sim_{CR} \subseteq \sim \subseteq \approx$. *Then* $\{(a,b) \in U \times U\colon\ a \sim \mathcal{L} b \sim\} = \{(a,b) \in U \times U\colon\ a \approx \mathcal{L} b \approx\}$ *if and only if* $\approx / \sim$ *is contained in* $\mathcal{L}^0$ *on* $U/\sim$.

Proof. "⇐". We only need to show $\{(a,b) \in U \times U\colon\ a \approx \mathcal{L} b \approx\} \subseteq \{(a,b) \in U \times U\colon\ a \sim \mathcal{L} b \sim\}$.

Assume that $a \approx \mathcal{L} b \approx$, then $(a, xb), (b, ya) \in \approx$ for some $x, y \in U$. This gives $(a \sim, xb \sim), (b \sim, ya \sim) \in \approx / \sim \subseteq \mathcal{L}^0$. Then $a \sim = uxb \sim, b \sim = vya \sim$ for some $u, v \in U$. Therefore $a \sim \mathcal{L} b \sim$.

"⇒". Let $(a \sim, b \sim) \in \approx / \sim$. Then $(a, b) \in \approx$. By the hypothesis, $a \sim \mathcal{L} b \sim$. So $\approx / \sim \subseteq \mathcal{L}$ and hence $\approx / \sim \subseteq \mathcal{L}^0$. ■

Let $U, V \in \mathcal{L}(CR)$. The Mal′cev product $U \circ V$ of U and V is defined by $U \circ V = \{S \in CR\colon$ there exists $\rho \in ConS$ such that (i) $(x\rho)^2 = x\rho$ then $x\rho \in U$, (ii) $S/\rho \in V\}$.

Lemma 3.5. *Let* $V \in \mathcal{L}(CR)$. *Then*

$$\{S \in CR\colon\ S/\tau \cap \mathcal{L}^0 \in V\} = LZ \circ V.$$

Proof. It follows directly from the definition of Mal′cev product. ■

Lemma 3.6. *Let $V \in \mathcal{L}(CR)$. If $SL \subseteq V$, then $LZ \circ V$ is a variety.*

Proof. It follows directly from [4, Theorem 5.1]. ■

Theorem 3.7. *Let $\sim \in FICU, \sim_{CR} \subseteq \sim, V = V(\sim)$ and $\overleftarrow{V} = V(\overleftarrow{\sim})$. Then*

$$\overleftarrow{V} = \{S \in CR\colon\ S/\tau \cap \mathcal{L}^0 \in V \vee SL\} = LZ \circ (V \vee SL)$$

Proof. Note first that $W = \{S \in CR\colon\ S/\tau \cap \mathcal{L}^0 \in V \vee SL\}$ forms a variety by Lemma 3.5 and Lemma 3.6. And note that $a(\sim \cap E)\ \mathcal{L} b(\sim \cap E)$ iff $(a^0, a^0b^0), (b^0, b^0a^0) \in \sim \cap E$, and that $a(\overleftarrow{\sim} \cap E)\mathcal{L}b(\overleftarrow{\sim} \cap E)$ iff $(a^0, a^0b^0), (b^0, b^0a^0) \in \overleftarrow{\sim} \cap E = \overleftarrow{\sim}$. So

$$\begin{gathered}\{(a,b) \in U \times U\colon\ a(\sim \cap E)\mathcal{L}b(\sim \cap E)\} \\ = \{(a,b) \in U \times U\colon\ (a^0, a^0b^0), (b^0, b^0a^0) \in \sim, E(a) = E(b)\}\end{gathered}$$

and

$$\begin{gathered}\{(a,b) \in U \times U\colon\ a\overleftarrow{\sim}\mathcal{L}b\overleftarrow{\sim}\} \\ = \{(a,b) \in U \times U\colon\ (a^0, a^0b^0), (b^0, b^0a^0) \in \overleftarrow{\sim}\}.\end{gathered}$$

Then by Theorem 3.2, $\{(a,b) \in U \times U\colon\ a(\sim \cap E)\mathcal{L}\ b(\sim \cap E)\} = \{(a,b) \in U \times U\colon\ a\overleftarrow{\sim}\mathcal{L}b\overleftarrow{\sim}\}$ since $\overleftarrow{\sim \cap E} = \overleftarrow{\sim}$. By Lemma 3.4, we have $\sim \cap E/\overleftarrow{\sim} \subseteq \mathcal{L}^0$. Again by Theorem 3.2, we have $\{a \in U\colon\ (a, a^0) \in \sim \cap E\} = \{a \in U\colon\ (a, a^0) \in \overleftarrow{\sim}\}$, then by Lemma 3.3, $\sim \cap E/\overleftarrow{\sim} \subseteq \tau$. Thus $\sim \cap E/\overleftarrow{\sim} \subseteq \tau \cap \mathcal{L}^0$.

Let $U' = U/\overleftarrow{\sim}$. Since $\sim \cap E/\overleftarrow{\sim} \subseteq \tau \cap \mathcal{L}^0$ and $U'/\sim \cap E/\overleftarrow{\sim} \simeq U/\sim \cap E$ which is a subdirect product of $U/\sim$ and U/E, it follows $U'/\sim \cap E \in V \vee SL$, and hence $U'/\tau \cap \mathcal{L}^0 \in V \vee SL$. This gives $U' \in W$. Therefore $\overleftarrow{V} \subseteq W$.

Conversely, since $W(SL \subseteq W)$ is a variety, $W = V(\approx)$ for some fully invariant congruence $\approx \in FICU$ and $\sim_{CR} \subseteq \approx \subseteq E$. Let $T = U/\approx$. Then the induced congruence $\sim \cap E/\approx$(clearly $\approx \subseteq \sim \cap E$) is the least $V(\sim \cap E)$-congruence on T, and therefore $\sim \cap E/\approx$ is the least $V \vee SL$-congruence on T since $V(\sim \cap E) = V(\sim) \vee SL = V \vee SL$. So $\sim \cap E/\approx$ is contained in $\tau \cap \mathcal{L}^0$ on T. By Lemma 3.3 and Lemma 3.4, we have that $\{a \in U\colon\ (a, a^0) \in \sim \cap E\} = \{a \in U\colon\ (a, a^0) \in \approx\}$ and $\{(ab) \in U \times U\colon\ a(\sim \cap E)\mathcal{L}b(\sim \cap E)\} = \{(a,b) \in U \times U\colon\ a \approx \mathcal{L}b \approx\}$. Then by Theorem 3.2, we have $\overleftarrow{\sim \cap E} = \overleftarrow{\approx}$, i.e. $\overleftarrow{\sim} = \overleftarrow{\approx}$, and so $\overleftarrow{\sim} \subseteq \approx$. Thus $W \subseteq \overleftarrow{V}$. Now the desired conclusion follows. ■

Theorem 3.8. $\overleftarrow{T} = LRB, \overleftarrow{G} = \overleftarrow{SLG} = LRO$.

Proof. By Theorem 3.7 and by the fact that $S \in LRB$ iff S is a semilattice of left zero bands, we have $\overleftarrow{T} = LRB$.

By the well-known fact that $SLG = G \vee SL$ [also see 10, Proposistion IV.1. 10], we have $\overleftarrow{SLG} = \overleftarrow{G \vee SL} = \overleftarrow{G}$. So we only need to show $\overleftarrow{SLG} = LRO$.

Let $S \in \overleftarrow{SLG}$. Then by Theorem 3.7, $S/\tau \cap \mathcal{L}^0 \in SLG$. Let $e, f \in E(S)$. Then $(ef, (ef)^0), (efe, (efe)^0) \in \tau \cap \mathcal{L}^0$, and so $ef, efe \in E(S)$. Since $(ef, efe) \in \tau \cap \mathcal{L}^0$, there exists $x \in S$ such that $ef = xefe$, and hence $(ef)(efe) = x(efe)^2 = xefe = ef$, i.e. $efe = ef$. Thus $S \in LRO$.

Conversely, let $S \in LRO$, then S is a semilattice Y of left groups $L_\alpha \times G_\alpha$. Let

$$\rho = \{((i,a),(j,a))\colon\ a \in G_\alpha, i, j \in L_\alpha, \alpha \in Y\}$$

It is easily checked that ρ is a congruence on S by Yamada Theorem [see 10, V.2], and that $\rho \subseteq \tau, \rho \subseteq \mathcal{L}$ which gives $\rho \subseteq \mathcal{L}^0$, and thus $\rho \subseteq \tau \cap \mathcal{L}^0$. Since S/ρ is a semilattice of groups, i.e. $S/\rho \in SLG$, it follows that $S/\tau \cap \mathcal{L}^0 \in SLG$. Thus $S \in \overleftarrow{SLG}$ by Theorem 3.7. ■

Theorem 3.9. *Let $V \in \mathcal{L}(CR)$. Then $V \vee SL = \overleftarrow{V} \cap \overrightarrow{V}$.*

Proof. Clearly $V \vee SL \subseteq \overleftarrow{V} \cap \overrightarrow{V}$. Conversely, let $S \in \overleftarrow{V} \cap \overrightarrow{V}$. Then by Theorem 3.7 and its dual, we have $S/\tau \cap \mathcal{L}^0 \in V \vee SL$ and $S/\tau \cap \mathcal{R}^0 \in V \vee SL$. Since $\tau \cap \mathcal{L}^0 \cap \mathcal{R}^0 = l_S, S$ is a subdirect product of $S/\tau \cap \mathcal{L}^0$ and $S/\tau \cap \mathcal{R}'$, and therefore $S \in V \vee SL$. ■

Theorem 3.10. $[SL, CR]$ *is a subdirect product of $\mathcal{L}(CR)/\rho_l$ and $\mathcal{L}(CR)/\rho_r$.*

Proof. Let φ_l be a map from $[SL, CR]$ onto $\mathcal{L}(CR)/\rho_l$ defined by

$$\varphi_l\colon\ V \to V\rho_l$$

and let φ_r a map defined dually. Then by Theorem 2.4 and its dual, $\varphi_l[\varphi_r]$ is complete homomorphism from $[SL, CR]$ onto $\mathcal{L}(CR)/\rho_l[\mathcal{L}(CR)/\rho_r]$. Let $V, W \in [SL, CR]$ be such that $V\varphi_l = W\varphi_l$ and $V\varphi_r = W\varphi_r$, i.e. $\overleftarrow{V} = \overleftarrow{W}, \overrightarrow{V} = \overrightarrow{W}$. Then by Theorem 3.9, we have that $V = \overleftarrow{V} \cap \overrightarrow{V} = \overleftarrow{W} \cap \overrightarrow{W} = W$. Thus $[SL, CR]$ is a subdirect product of $\mathcal{L}(CR)/\rho_l$ and $\mathcal{L}(CR)/\rho_r$. ■

Let $\sim \in FICU$, by $\overset{\sim}{\leftarrow}$ $[\overset{\sim}{\rightarrow}]$ we denote the maximum f. i. congruence in the $\sim$ $-$class of $\rho_l[\rho_r]$ (they must exist $\overset{\sim}{\leftarrow}$ and $\overset{\sim}{\rightarrow}$ by Theorem 2.4). Let $\overset{V}{\leftarrow} = V(\overset{\sim}{\leftarrow}), \overset{V}{\rightarrow} = V(\overset{\sim}{\rightarrow})$ (where $V = V(\sim)$).

Let $V = V(u_i = v_i\colon\ i \in I)$. In order to find the identities that determine $\overleftarrow{V}$, we need the following Lemmas.

Lemma 3.11. *If $u, v \in U, x, y \notin E(u) \cup E(v), e, f \in U, E(u) \cup E(v) \subseteq E(e) \cap E(f)$, then the identities $exu = exv$ and $fyu = fyv$ are equivalent in the variety CR.*

Proof. Assume that $\sim\in \mathit{FICU}, \sim_{CR}\subseteq\sim, exu\sim exv$. Substituting here fy for all variables different from those of $E(u)\cup E(v)$, we get $e'.fy.u\sim e'.fy.v$ for some $e'\in U, E(e')\subseteq E(f)\cup\{y\}$. Multiplying the last expression by $fy(e'fy)^{-1}$ on the left, we get $fy(e'fy)^0u\sim fy(e'fy)^0v$. By the dual of Lemma 2.1(2), we have $fyu\sim fyv$.

Thus the identity $exu=exv$ implies the identity $fyu\sim fyv$ in the variety CR, and the proof of the converse is analogous. ■

Lemma 3.12 [12, Theorem 1.5(2), (5) and (6)]**.** *Let* $\sim=\sim(u_i=v_i\colon\ i\in I)$. *Then*

(1) $\sim_0=\sim\subseteq E$ *iff* $(0^k(u_i),0^k(v_i))\in E\cap\sim$ *for all* $i\in I, k\geq 0$.
(2) $\sim_0\subseteq E$ *iff* $E(0^k(u_i))=E(0^k(v_i))$ *for all* $i\in I, k\geq 0$.
(3) *if* $\sim_0\subseteq E$, *then* $\sim_0=\sim(0^k(u_i)=0^k(v_i)\colon\ i\in I, k\geq 0)$.

Theorem 3.13. *Let* $V=V(u_i=v_i\colon\ i\in I)$ *and write* $\sim$ *instead of* $\sim_v$. *Then*

(1) *for any* $a,b\in U, a\overset{\sim}{\leftarrow}b$ *if and only if* $exa\sim exb$ *for a variable* x *and* $e\in U$ *such that* $x\notin E(a)\cup E(b)\subseteq E(e)$.

(2) *the followings are equivalent:*

  (i) $\overleftarrow{V}=V$.
  (ii) $\sim\subseteq E, \sim_0=\sim$.
  (iii) $(0^k(u_i),0^k(v_i))\in E\cap\sim$ *for all* $i\in I, k\geq 0$.

(3) $\overleftarrow{V}=V(e_ix_iu_i=e_ix_iv_i\colon\ i\in I)$ *where, for all* $i\in I$, *a variable* $x_i\notin E(u_i)\cup E(v_i)\subseteq E(e_i)$.

(4) $(\overleftarrow{V})_0=\overleftarrow{V}$.

Proof. To prove (1), denote

$\approx=\{(a,b)\in U\times U\colon\ exa\sim exb$ for a variable x and a word e such that $x\notin E(a)\cup E(b)\subseteq E(e)\}$. First of all we use Lemma 2.2 and the remark after this Lemma to prove that $\approx\in \mathit{FICU}$. Clearly, the relation $\approx$ is reflexive and symmetric, and its transitivity follows from Lemma 3.11.

If $a\approx b, a,b,c\in U$, then substituting cyc for $x(y\notin E(a)\cup E(b)\cup E(c))$ in $exa\sim exb$, we get $e'cyca\sim e'cycb$ for some $e'\in U, E(a)\cup E(b)\subseteq E(e')\subseteq E(e'c)$. Thus $ca\approx cb$. Substituting cy for $x(y\notin E(a)\cup E(b)\cup E(c))$ in $exa\sim exb$, we get $e''cya\sim e''cyb$ for some $e''\in U, E(a)\cup E(b)\subseteq E(e'')$. Then multiplying the last expression by c on the right, we get $e''cyac\sim e''cybc$. Thus $ac\approx bc$.

Suppose now that $a\approx b, a,b,p_1,p_2,\ldots\in U$. Substituting q_i for $x_i(i=1,2,\ldots)$, where

$$q_i=\begin{cases} y, & \text{if } x_i=x\\ p_i, & \text{otherwise}, y\notin E(p_1)\cup E(p_2)\cup\cdots\end{cases}$$

in $exa \sim exb$, we get $e(q_1,\ldots)ya(p_1,\ldots) \sim e(q_1,\ldots)yb(p_1,\ldots)$ which yields $a(p_1,\ldots) \approx b(p_1,\ldots)$. Therefore $\approx \in FICU, \sim_{CR} \subseteq \approx$.

In view of Theorem 3.2, it suffices to show that $\approx$ is the greatest f. i. congruence on U with the following properties:

$$\sim_{CR} \subseteq \approx, \{a \in U\colon\ a^0 \approx a\} = \{a \in U\colon\ a^0 \sim a\} \tag{$*$}$$

$$\begin{aligned}&\{(a,b) \in U \times U : (a^0, a^0b^0), (b^0, b^0a^0) \in \approx, E(a) = E(b)\}\\ &\quad = \{(a,b) \in U \times U : (a^0, a^0b^0), (b^0, b^0a^0) \in \sim, E(a) = E(b)\}\end{aligned} \tag{**}$$

Now, $\sim \subseteq \approx$ gives $\{a \in U\colon\ a^0 \sim a\} \subseteq \{a \in U\colon\ a^0 \approx a\}$, and if $a^0 \approx a$, then $a^0xa^0 \sim a^0xa$ for $x \notin E(a)$ by Lemma 3.11. Substituting here a^0 for x, we get $a^0 \sim a$. Thus $\approx$ has the property (*). Also $\sim \subseteq \approx$ gives

$$\begin{aligned}&\{(a,b) \in U \times U : (a^0, a^0b^0), (b^0, b^0a^0) \in \sim, E(a) = E(b)\}\\ &\quad \subseteq \{(a,b) \in U \times U : (a^0, a^0b^0), (b^0, b^0a^0) \in \approx, E(a) = E(b)\},\end{aligned}$$

and if $a^0 \approx a^0b^0, b^0 \approx b^0a^0, E(a) = E(b)$, then $a^0xa^0 \sim a^0xa^0b^0$ for $x \notin E(a)$, and $b^0xb^0 \sim b^0xb^0a^0$ for $x \notin E(a)$, by Lemma 3.11. Substituting here a^0, b^0 for x respectively, we get $a^0 \sim a^0b^0$ and $b^0 \sim b^0a^0$. Thus $\approx$ has the property (**).

Assume now that $\sim' \in FICU$, such that $\sim'$ satisfies (*) and (**) and $a \sim' b$. We want to show that $a \approx b$. Choose a variable x and a word e such that $x \notin E(a) \cup E(b) \subseteq E(e)$. Multiplying $exa \sim' exb$ by $(exb)^{-1}$ on the left, we get

$$(exb)^{-1}exa \sim' (exb)^0$$

which gives

$$(exb)^0(exa)^0 \sim' (exb)^{-1}(exa)(exa)^0 \sim_{CR} (exb)^{-1}(exa) \sim' (exb)^0,$$

similarly we have

$$(exa)^0(exb)^0 \sim' (exa)^0,$$

and so

$(exb)^0 \sim (exb)^0(exa)^0$ by the hypothesis. This yields

$$exb \sim (exb)(exa)^0 \tag{1}$$

Multiplying $exa \sim' exb$ by $(exa)^{-1}$ on the right, we get

$$(exb)(exa)^{-1} \sim' (exa)^0$$

which gives

$$((exb)(exa)^{-1})^0 \sim' (exb)(exa)^{-1}.$$

Then by the hypothesis, we have

$$((exb)(exa)^{-1})^0 \sim (exb)(exa)^{-1}.$$

By Lemma 2.1(5), we have $((exb)(exa)^{-1})^0 \sim_{CR} (exa)^0$. Thus

$$(exb)(exa)^{-1} \sim (exa)^0 \tag{2}$$

Then we have

$$\begin{aligned} exb &\sim (exb)(exa)^0 \sim_{CR} (exb)(exa)^{-1}(exa) \quad by(1) \\ &\sim (exa)^0(exa) \sim_{CR} exa \quad by(2). \end{aligned}$$

Therefore $a \approx b$.

Now we consider the "$(i) \Rightarrow (ii)$" part of (2). Clearly $\sim \subseteq E$ and $\sim \subseteq \sim_0$. We only need to show that $\sim_0 \subseteq \sim$. Let $a \sim_0 b$. Then there exist $c, d \in U$ such that $c \sim d, 0(c) = a$ and $0(d) = b$. Since $\sim = \overleftarrow{\sim}$, it follows that $c \overleftarrow{\sim} b$ which gives $0(c) \overleftarrow{\sim} 0(d)$. So $a \sim b$.

To prove the opposite implication, we use induction with respect to $C(a)$ to show that

$$\sim \subseteq E, \sim_0 \subseteq \sim, a \sim b \text{ implies } a \overleftarrow{\sim} b.$$

Indeed, if $C(a) = 1, a \sim b$ then $a \overleftarrow{\sim} b$ since $E(a) = E(b)$. Suppose now that $C(a) = n \geq 2, a \sim b, E(a) = E(b)$. Then $(0(a), 0(b)) \in \sim_0 \subseteq \sim$ and $(0(a), 0(b)) \in \overleftarrow{\sim}$ by the inductive assumption. Thus $a \overleftarrow{\sim} b$.

The equivalence of (ii) and (iii) of (2) is an immediate consequence of Lemma 3.12(1).

To establish (3), denote the variety on the right by W. Then $V \subseteq W, \overleftarrow{W} = W$ by(2), $\sim \subseteq \xleftarrow{\sim_W}$ by (1), which together gives $W = \overleftarrow{V}$.

Assertion (4) follows from (3) and Lemma 3.12(2), (3). ∎

Theorem 3.14. *Let $V \in \mathcal{L}(CR)$. Then*

$$V \vee SL = \xleftarrow{V} \vee \xrightarrow{V} \vee SL.$$

Proof. Denote the variety on the right by W. Then, obviously $W \subseteq V \vee SL$ which gives $\overleftarrow{V} \subseteq \overleftarrow{V \vee SL} = \overleftarrow{V}$ by Theorem 2.4. But $\overleftarrow{W} \supseteq \overleftarrow{(\xleftarrow{V} \vee SL)} = \overleftarrow{(V \vee SL)}$ also by Theorem 2.4. So we have $\overleftarrow{W} = \overleftarrow{V \vee SL}$. Similarly $\overrightarrow{W} = \overrightarrow{V \vee SL}$. Thus $W = V \vee SL$ by Theorem 3.9. ∎

Theorem 3.15. *Let $V_i \in \mathcal{L}(CR)$ for $i \in I, I \neq \emptyset$, and write $\sim_i$ briefly for $\sim_{V_i}$ $(i \in I)$. Then*

(1) $\xleftarrow{\cap \sim_i} = \cap \xleftarrow{\sim_i}, \xleftarrow{\vee V_i} = \vee \xleftarrow{V_i}$.

(2) $\xleftarrow{\vee \sim_i} \supseteq \vee \xleftarrow{\sim_i}, \xleftarrow{\cap V_i} \subseteq \cap \xleftarrow{V_i}$.

Proof. Since $\overleftarrow{(\xleftarrow{\cap\sim_i})} = \overleftarrow{\cap\sim_i} = \cap\overleftarrow{\sim}_i = \cap\overleftarrow{\xleftarrow{\sim_i}} = \overleftarrow{\cap\xleftarrow{\sim_i}}$ by Theorem 2.14, it follows that $\xleftarrow{\cap\sim_i}\subseteq \cap \xleftarrow{\sim_i}$. To prove the opposite inclusion, let $(a,b) \in \cap\overleftarrow{\sim}_i$, then $a\overleftarrow{\sim}_i b$ for all $i \in I$. Choose a variable x and a word e such that $x \notin E(a) \cup E(b) \subseteq E(e)$, we have $exa \sim_i exb$ for all $i \in I$ by Theorem 3.13(1), i.e., $(exa, exb) \in \cap \sim_i$. Again by Theorem 3.13(1), we have $(a,b) \in\xleftarrow{\cap\sim_i}$.

The second one is its reformulations.

(2) It is immediate. ■

Example 3.16. Let $V_1[V_2]$ be the varieties of completely simple semigroups over abelian subgroups of exponent 2 [3]. Then by [12, Theorem 2.6], $\underline{V_1} = V_1, \underline{V_2} = V_2$. Since $V_1 \cap V_2 = ReB$, it follows that $\underline{V_1 \cap V_2} = T$ by [1, Theorem 2]. Thus

$$\underline{V_1} \cap \underline{V_2} = V_1 \cap V_2 = ReB \neq T = \underline{V_1 \cap V_2}.$$

Since $\underline{V} \subseteq\xleftarrow{V}$ for every $V \in \mathcal{L}(CR)$, it follows that $\xleftarrow{V_1}= V_1$ and $\xleftarrow{V_2}= V_2$. It is easy to check that $\xleftarrow{ReB}= RZ$. Thus

$$\xleftarrow{V_1} \cap \xleftarrow{V_2}= ReB \neq RZ =\xleftarrow{V_1\cap V_2}.$$

Remark 3.17. The duals of the Lemmas and theorems in this section also hold.

4. Semigroups generated by certain operators on varieties of completely regular semigroups

For a set A of operators on $\mathcal{L}(CR)$, we denote by $[A]$ the subsemigroup of the full transformation semigroup on $\mathcal{L}(CR)$ generated by A. The free semigroup on a nonempty set X is denoted by X^+. A semigroup given by generators G and relations R is denoted by $\langle G|R\rangle$. For a semigroup S, S^1 (respectively S^0) stands for S with an identity (respectively zero) adjoined.

In this section we will determine various semigroups of operators in terms of generators and relations. We shall require some additional notation.

Let A be a set. An element $w = a_1a_2\dots a_n \in A^+$ will be called distinguished if $a_i \neq a_{i+1}$ for $i = 1,\dots,n-1$. The set of distinguished elements in $\{a,b,c,\dots\}^+$ will be denoted by Δ_{abc}. If $v \in A^+$ and we wish to emphasize the variables $x_1,\dots,x_n \in A$ that appear in v, then we shall write $v(x_1,\dots,x_n)$. If S is a semigroup and $s_1,\dots,s_n \in S$, then we shall denote by $v(s_1,\dots,s_n)$ the element obtained from v by substituting s_i for $x_i, i = 1,\dots,n$.

An operator $x \to x^*$ on a lattice L is an upper [lower] closure operator if

$$x \leq y \Leftarrow x^* \leq y^*, x \leq x^*[x^* \leq x], (x^*)^* = x^*(x, y \in L).$$

The operator L is defined on $\mathcal{L}(CR)$ as follows:

$$VL = \{S \in CR\colon\ eSe \in V \text{ for all } e \in E(S)\}(V \in \mathcal{L}(CR)).$$

The four operators K, T, T_1 and T_r can be found in [9]. Now we introduce their alternative descriptions.

In [12], the relations τ_0, τ_1 and τ on $\mathcal{L}(CR)$ are defined as follows: $V\tau_0 W \Leftrightarrow (\sim_V)_0 = (\sim_W)_0$ (that is iff $V_0 = W_0$ and ($SL \subseteq V$ iff $SL \subseteq W$)).

$V\tau_1 W \Leftrightarrow (\sim_V)_1 = (\sim_W)_1$ (that is iff $V_1 = W_1$ and ($SL \subseteq V$ iff $SL \subseteq W$)).

$V\tau W \Leftrightarrow V_0 = W_0, V_1 = W_1$ and ($SL \subseteq V$ iff $SL \subseteq W$).

Let V^0 be the greatest variety in τ_0-class of V (it exists by [2, Theorem 1.6 (1)]), define V^1 dually, and let V^- and V^+ be the smallest and greatest variety, respectively, in the $\tau-$class of V.

Lemma 4.1. *Let* $V \in \mathcal{L}(CR)$*.*

(1) $VK = \overline{V} = RB \circ (V \vee SL) = B \circ (V \vee SL) = \{S \in CR\colon\ S/\tau \in V \vee SL\} = \{S \in CR\colon\ S/\tau \cap \mathcal{D} \in V \vee SL\}$.

(2) $VT = V^+ = G \circ V = \{S \in CR\colon\ S/\mu \in V\}$.

(3) $VT_l = V^1 = LG \circ V = \{S \in CR\colon\ S/\mathcal{L}^0 \in V\}$.

(4) $VT_r = V^0 = RG \circ V = \{S \in CR\colon\ S/\mathcal{R}^0 \in V\}$.

Proof. See [9, Lemma 4.3]. The equalities $VK = \overline{V}$, $VT = V^+$, $VT_l = V^1$ and $VT_r = V^0$ can be found in [13, section 9]. ■

Remark 4.1′. The operators L, K, T, T_l and T_r are all upper closure operators on $\mathcal{L}(CR)$. And they all respect intersection. Moreover, the operator K respects join. [see 9 and 11].

For convenience, we denote $\overleftarrow{(\ )}$ and $\overrightarrow{(\ )}$ by T_l^* and T_r^* respectively.

By the following lemma, we reveal the connection between the operator $\overleftarrow{(\ )}$ and the operators $\overline{(\ )}$ and $()^1$.

Lemma 4.2. *Let* $V \in \mathcal{L}(CR)$*. Then*

$$\overleftarrow{V} = (V \vee SL)^1 \cap \overline{V}.$$

Proof. Let $S \in \overleftarrow{V}$. Then by Theorem 3.7, $S/\tau \cap \mathcal{L}^0 \in V \vee SL$ which gives $S/\tau \in V \vee SL$ and $S/\mathcal{L}^0 \in V \vee SL$. Then by Lemma 4.1, $S \in (V \vee SL)^1 \cap \overline{V}$. Thus $\overleftarrow{V} \subseteq (V \vee SL)^1 \cap \overline{V}$. To prove the opposite inclusion, let $S \in (V \vee SL)^1 \cap \overline{V}$, then by Lemma 4.1, we have that $S/\mathcal{L}^0 \in V \vee SL$ and $S/\tau \in V \vee SL$. Since $S/\tau \cap \mathcal{L}^0$ is a subdirect product of S/τ and $S/\mathcal{L}^0$, it follows that $S/\tau \cap \mathcal{L}^0 \in V \vee SL$. Again by Theorem 3.7, we have $S \in \overleftarrow{V}$, as required. ■

Lemma 4.3 [9, Section 5, 6]**.** *L commutes with K, T, T_l and T_r.*

Lemma 4.4 [9, Section 5, 6]**.** *Let* $V \in \mathcal{L}(CR)$*. Then* $(V \vee SL)L = VL \vee SL$*.*

Theorem 4.5. $LT_l^* = T_l^* L$.

Proof. Let $V \in \mathcal{L}(CR)$. Then

$$\begin{aligned}
VLT_l^* &= \overleftarrow{VL} = (VL \vee SL)^1 \cap \overline{VL} \quad \text{by Lemma 4.2}\\
&= ((V \vee SL)L)^1 \cap \overline{VL} \quad \text{by Lemma 4.4}\\
&= (V \vee SL)^1 L \cap \overline{VL} \quad \text{by Lemma 4.3}\\
&= ((V \vee SL)^1 \cap \overline{V})L \quad \text{since } L \text{ respects intersection}\\
&= VT_l^* L \quad \text{by Lemma 4.2}
\end{aligned}$$

Thus $LT_l^* = T_l^* L$. ∎

Now we study the semigroup generated by T_l^* and T_r^*. We need some Lemmas.

Lemma 4.6 [11, Lemma 4]**.** *Let $V \in \mathcal{L}(CR)$. Then $SL \not\subseteq V$ iff $V \subseteq CS$.*

Lemma 4.7 [12, Theorem 2.4; 5, Proposition 8.2]**.** *Let $V \in \mathcal{L}(\mathrm{CR})$. Then*
(1) $(\overline{V})_0 = (\overline{V})_1 = \overline{V}$.
(2) *$V_0 = V_1 = V, SL \subseteq V$ iff $V = \overline{V}$.*
(3) *If $RB \subseteq V$, then $\underline{V^+} = V^+$.*

Lemma 4.8. *If V is a proper subvariety of $\overline{W}$, then so are $V \vee SL, \overleftarrow{V}$ and $\overrightarrow{V}$.*

Proof. First of all we show that $V \vee SL$ is a proper subvariety of $\overline{W}$.

The case when $SL \subseteq V$ is trivial. So we only need to consider the case when $SL \not\subseteq V$, i.e., $V \subseteq CS$ by Lemma 4.6. Suppose that $V \vee SL = \overline{W}$. Then

$$\begin{aligned}
\overline{W} &= (\overline{W})_0 \quad \text{by Lemma 4.7 (1)}\\
&= (V \vee SL)_0 \quad \text{by the hypothesis}\\
&= V_0 \vee SL_0 \quad \text{by [12, Theorem 1.6]}\\
&\subseteq LZ \vee SL \quad \text{a fact}
\end{aligned}$$

But $B \subseteq \overline{W}$, a contradiction, so we have $V \vee SL \subset \overline{W}$.

Suppose now that $\overleftarrow{V} = \overline{W}$. By Lemma 4.2, we have $V \subseteq \overleftarrow{V} \subseteq \overline{V}$. Since $\overleftarrow{V} = \overline{W}$, it follows that $\overleftarrow{V} = \overline{V} = \overline{W}$. Then by Lemma 4.2, $(V \vee SL)^1 \supseteq \overline{V}$. By Lemma 4.7, we have $V \vee SL \supseteq (V \vee SL)_1 \supseteq \overline{V}$, and so $V \vee SL = \overline{V} = \overline{W}$. By the above discussion, we get a contradiction. Thus $\overleftarrow{V} \subset \overline{W}$.

Similarly we have $\overrightarrow{V} \subset \overline{W}$. ∎

Lemma 4.9 [12, Theorem 1.6]**.** *Let $V_i \in \mathcal{L}(CR)$ for $i \in I, I \neq \emptyset$. Then*
(1) $(\vee V_i)_0 = \vee (V_i)_0, (\vee V_i)_1 = \vee (V_i)_1$,
(2) $(\cap V_i)_0 = \cap (V_i)_0, (\cap V_i)_1 = \cap (V_i)_1$.

Lemma 4.10 [12, Theorem 1.7]. *Let* $V, V_i \in \mathcal{L}(CR)$, *for* $i \in I$. *Then*

$$(V^0)_0 = V_0, (V^0)_1 = \begin{cases} RZ & \text{if } V \subseteq CS \\ V^0 & \text{otherwise.} \end{cases}$$

Lemma 4.11. *Let* $V \in \mathcal{L}(CR)$. *Then*

$$(VT_l^*T_r^*)_0 = VT_l^* \text{ and } (VT_l^*T_r^*)_1 = VT_l^*T_r^*.$$

Theorem 4.12. *Let* $V \in \mathcal{L}(CR), V \neq \overline{V}$. *Then*

$$VT_l^*T_r^* \subset V(T_l^*T_r^*)^2 \subset \cdots \subset \overline{V} \text{ and } \vee V(T_l^*T_r^*)^n = \overline{V}.$$

Proof. It follows immediately from Lemma 4.8 that $V(T_l^*T_r^*)^n$ is a proper subvariety of $\overline{V}$ for all $n \in Z^+$.

Let W be the union of $V(T_l^*T_r^*)^n$. Then by Lemma 4.9(1), W_0 is the union of $(V(T_l^*T_r^*)^n)_0$ which is equal to $V(T_l^*T_r^*)^{n-1})T_l^* \supseteq V(T_l^*T_r^*)^{n-1}$. So $W_0 = W$. Similarly we have $W_1 = W$. Then by Lemma 4.7(2) and by the fact $SL \subseteq W$, we have $W = \overline{W}$. Since $V \subseteq W \subseteq \overline{V}$, it follows that $W = \overline{V}$. It now follows that $V(T_l^*T_r^*)^n \subset V(T_l^*T_r^*)^{n+1}$ for all $n \in Z^+$. ■

Theorem 4.13. $[T_l^*, T_r^*] \simeq \langle t_l, t_r\colon\ t_l^2 = t_l, t_r^2 = t_r\rangle$.

Proof. Let $\varphi\colon \{t_l, t_r\}^+ \to [T_l^*, T_r^*]$ be the epimorphism defined by $t_l\varphi = T_l^*, t_r\varphi = T_r^*$. Let $\rho = \varphi \circ \varphi^{-1}$ and σ be the congruence on $\{t_l, t_r\}^+$ genrated by the relation $\{(t_l^2, t_l), (t_r^2, t_r)\}$. Since $T_l^{*2} = T_l^*$ and $T_r^{*2} = T_r^*$, it follows that $\sigma \subseteq \rho$. To establish the reverse inclusion, let $p, q \in \{t_l, t_r\}^+$ and $p\rho q$. Using the relation defining σ, there exist $v, w \in \Delta t_l t_r$ such that $p\sigma v$ and $q\sigma w$. Since $\sigma \subseteq \rho$, we have $v\rho w$. Thus $v(T_l^*, T_r^*) = w(T_l^*, T_r^*)$. Since $v \in \Delta t_l t_r$, we have $v(T_l^*, T_r^*) = (T_l^*T_r^*)^m$, or $T_r^*(T_l^*T_r^*)^m$ or $(T_l^*T_r^*)^m T_l^*$, or $T_r^*(T_l^*T_r^*)^m T_l^*$ for some $m \in Z^+$. Similarly, $w(T_l^*, T_r^*) = (T_l^*T_r^*)^n$, or $T_r^*(T_l^*T_r^*)^n$, or $(T_l^*T_r^*)^n T_l^*$, or $T_l^*(T_l^*T_r^*)^n T_l^*$ for some $n \in Z^+$. Thus there are 16 cases.

Case 1. Suppose that $v(T_l^*, T_r^*) = (T_l^*T_r^*)^m, w(T_l^*, T_r^*) = (T_l^*T_r^*)^n$, and $v(T_l^*, T_r^*) = w(T_l^*, T_r^*)$. Then $SL(T_l^*T_r^*)^m = SL(T_l^*T_r^*)^n$. By Theorem 4.12, we have $m = n$. Thus $v = w$.

Case 2. Suppose that $v(T_l^*, T_r^*) = (T_l^*T_r^*)^m, w(T_l^*, T_r^*) = T_r^*(T_l^*T_r^*)^n$ and $v(T_l^*, T_r^*) = w(T_l^*, T_r^*)$. Then $(SLT_l^*)(T_l^*T_r^*)^m = (SLT_l^*)T_r^*(T_l^*T_r^*)^n$, i.e., $SL(T_l^*T_r^*)^m = SL(T_l^*T_r^*)^{n+1}$. Also we have $SL(T_l^*T_r^*)(T_l^*T_r^*)^m = SL(T_l^*T_r^*) T_r^*(T_l^*T_r^*)^n$, i.e., $SL(T_l^*T_r^*)^{m+1} = SL(T_l^*T_r^*)^{n+1}$. Again by Theorem 4.12, we have $m = n+1$ and $m+1 = n+1$, a contradiction.

Case 3. Suppose that $v(T_l^*, T_r^*) = (T_l^*T_r^*)^m, w(T_l^*, T_r^*) = (T_l^*T_r^*)^n T_l^*$ and $v(T_l^*, T_r^*) = w(T_l^*, T_r^*)$. Multiplying the last expression by $T_l^*T_r^*$ and T_r^* on the right respectively, we get $(T_l^*T_r^*)^{m+1} = (T_l^*T_r^*)^{n+1}$ and $(T_l^*T_r^*)^m = (T_l^*T_r^*)^{n+1}$. Using the same method in Case 2, we get a contradiction.

Case 4. Suppose that $v(T_l^*, T_r^*) = (T_l^*T_r^*)^m, w(T_l^*, T_r^*) = T_r^*(T_l^*T_r^*)^n T_l^*$, and $v(T_l^*, T_r^*) = w(T_l^*, T_r^*)$. Multiplying the last expression by T_r^* on the right, we get $(T_l^*T_r^*)^m = T_r^*(T_l^*T_r^*)^{n+1}$. Then we get a contradiction by using the proof in Case 2.

Other cases are similar. So we have $v = w$. Thus $p\sigma v = w\sigma q$ so that $p\sigma q$ and $\sigma \subseteq \rho$. Thus $\sigma = \rho$. ■

Now we consider the semigroup $[K, T_l^*]$ which is an immediate consequence of the following lemma.

Lemma 4.14. *Let $V \in \mathcal{L}(CR)$. Then $\overleftarrow{\overline{V}} = \overline{V}, \overline{\overleftarrow{V}} = \overline{V}$.*

Proof. By Lemma 4.2, we have

$$\overleftarrow{\overline{V}} = (\overline{V} \vee SL)^1 \cap \overline{\overline{V}} = (\overline{V})^1 \cap \overline{V} = \overline{V}$$

and

$$\overline{\overleftarrow{V}} = \overline{(V \vee SL)^1 \cap \overline{V}} = \overline{(V \vee SL)^1} \cap \overline{V} = \overline{V}$$ ■

Theorem 4.15. $[K, T_l^*] \simeq \langle t_l | t_l^2 = t_l \rangle^0$.

Now we consider the semigroup $[T_l^*, T]$. We need some lemmas.

Lemma 4.16 [6, Lemma 7.7]**.** $TT_l = T_lT = T_l$.

Lemma 4.17. *Let $V \in \mathcal{L}(CR)$. Then*

$$\vee V(KT)^n = CR \text{ and } F_n \in V(KT)^n.$$

Proof. See [9, Theorem 4.6(i)] and [6, Lemma 5.13]. ■

Lemma 4.18. *Let $V, W \in \mathcal{L}(CR)$. Then*

$$W = \vee(V(KT)^n \cap W).$$

Proof. It suffices to show that every finitely generated completely regular semigroup S in W belongs to $\vee(V(KT)^n \cap W)$. Let $S \in W$ be such that it is generated by $\{x_1, x_2, \ldots, x_n\}$. Then $S \in F_n \cap W \subseteq V(KT)^n \cap W$ by Lemma 4.17, and the desired conclusion follows. ■

Lemma 4.19. *Let $V \in \mathcal{L}(CR), RB \subseteq V$. If V is a proper subvariety of $(W \vee SL)^1$, then so are $\overleftarrow{V}$ and V^+.*

Proof. Suppose $\overleftarrow{V} = (W \vee SL)^1$. Then $(W \vee SL)^1 = (V \vee SL)^1 \cap \overline{V}$ by Lemma 4.2. This gives $(W \vee SL)^1 \subseteq (V \vee SL)^1$ and $(W \vee SL)^1 \subseteq \overline{V}$. And so $(V \vee SL)^1 = (W \vee SL)^1$ and $(V \vee SL)^1 \subseteq \overline{V}$. Then by Lemma 4.7(3) and the fact that $((V \vee SL)^1)^+ = (V \vee SL)^1$, we have $(V \vee SL)^1 = \underline{(V \vee SL)^1} \subseteq \underline{V} \subseteq V$ which gives $(V \vee SL)^1 = V$, and hence $V = (W \vee SL)^1$, a contradiction.

Now suppose that $(W \vee SL)^1 = V^+ = V^0 \cap V^1$. Then $(W \vee SL)^1 \subseteq V^1$ yields $(W \vee SL)^1 = V^1$. So $V^1 \subseteq V^0$. Then by Lemma 4.10 (1) and its dual, we have $V \subseteq V^1 = (V^1)_0 \subseteq (V^0)_0 = V_0 \subseteq V$ which gives $V = V^1$ and hence $V = (W \vee SL)^1$, a contradiction. ■

Theorem 4.20. *Let $V \in \mathcal{L}(CR), RB \subseteq V$ and $V \neq V^1$. Then*

$$V \subset VT_l^*T \subset V(T_l^*T)^2 \subset \cdots \subset V^1 \text{ and } \vee V(T_l^*T)^n = V^1.$$

Proof. First we shall show that $V(T_l^*T)^n = V(KT)^n \cap V^1$. We use induction with respect to n. If $n = 0$, then there is nothing to prove.

Note that the operators T and K respect intersaction.

Suppose that the above equality holds for $n = k$ and let $n = k+1$. We have

$$\begin{aligned} V(T_l^*T)^{k+1} &= V(T_l^*T)^k T_l^*T \\ &= ((V(T_l^*T)^k)^1 \cap V(T_l^*T)^k K)T \text{ by Lemma 4.2} \\ &= ((V(KT)^k \cap V^1)^1 \cap (V(KT)^k \cap V^1)K)T \\ &= (V^1 \cap (V(KT)^k \cap V^1)K)T \\ &= (V^1 \cap V(KT)^k K \cap V^1 K)T \text{ since } K \text{ respects intersaction} \\ &= (V^1 \cap V(KT)^k K)T \\ &= V^1 T \cap V(KT)^{k+1} \text{ since } T \text{ respects intersaction} \\ &= V^1 \cap V(KT)^{k+1} \text{ by Lemma 4.16} \end{aligned}$$

Thus the equality $V(T_l^*T)^n = V(KT)^n \cap V^1$ holds for all $n \in Z^+$.

By Lemma 4.18 and by the above discussion, we have $\vee V(T_l^*T)^n = V^1$. Now the inclusion $V(T_l^*T)^n \subset V(T_l^*T)^{n+1}$ follows from Lemma 4.19. ■

Theorem 4.21. $[T_l^*, T] = \langle t_l, t | t_l^2 = t_l, t^2 = t \rangle$.

Proof. Let φ: $\{t_l, t\}^+ \to [T_l^*, T]$ be the epimorphism defined by $t_1\varphi = T_l^*, t\varphi = T$. Let $\rho = \varphi \circ \varphi^{-1}$ and σ be the congruence on $\{t_l, t\}^+$ generated by the relation $\{(t_l^2, t_l\}, (t^2, t)\}$. Since $T_l^{*2} = T_l^*, T^2 = T$, it follows that $\sigma \subseteq \rho$. To establish the reverse inclusion, let $p, q \in \{t_l, t\}^+$ and $p\rho q$. Using the relation defining σ, there exist $v, w \in \Delta t_l t$ such that $p\sigma v$ and $q\sigma w$. Since $\sigma \subseteq \rho$, we have $v\rho w$. Thus $v(T_l^*, T) = w(T_l^*, T)$. Using the same method in the proof of Theorem 4.13, the conclusion follows. ■

5. The join of V_1 and V_2 which satisfy $V_1 \subseteq \overrightarrow{V_2}$ and $V_2 \subseteq \overleftarrow{V_1}$

In [7], F. Pastijn proved that the lattice $\mathcal{L}(CR)$ is an argusian lattice, and therefore it is a modular lattice. Since $\mathcal{L}(CR)$ is not a distributive lattice, the equality

$$W \cap (V_1 \vee V_2) = (W \cap V_1) \vee (W \cap V_2)$$

doesn't hold in general. In this section, we find some conditions under which the above equality holds.

In [14], the join of V_1 and V_2 which satisfy $V_1 \subseteq \overline{V_2}$ and $V_2 \subseteq {V_1}^+$ were considered, and the following results were obtained:

Theorem [14, Theorem 6]. *Let* $V_1, V_2 \in \mathcal{L}(CR)$. *Then the equality* $V_1 \vee V_2 = \overline{V_2} \cap {V_1}^+$ *holds if and only if* $V_1 \subseteq \overline{V_2}, V_2 \subseteq {V_1}^+$.

Theorem [14, Theorem 7]. *Let* $V_1, V_2 \in \mathcal{L}(CR), SL \subseteq V_2, V_1 = [u_\alpha = v_\alpha \colon \alpha \in A], V_2 = [s_\beta = t_\beta \colon \beta \in B]$ *and* $A \cap B = \emptyset$. *If* $V_1 \subseteq \overline{V_2}, V_2 \subseteq {V_1}^+$ *and* $S \in CR$. *Then the following conditions are equivalent.*

(1) $S \in V_1 \vee V_2$.

(2) S *is a subdirect product of a semigoup* $S_1 \in V_1$ *and a semigroup* $S_2 \in V_2$.

(3) S *satisfies the identities*

$$e_\beta x_\beta s_\beta x_\beta e_\beta = e_\beta x_\beta t_\beta x_\beta e_\beta (\beta \in B); u_\alpha^0 = v_\alpha^0, (x u_\alpha y)^0 = (x v_\alpha y)^0 (\alpha \in A).$$

Where, for all $\beta \in B$, *a variable* x_β *and* $e_\beta \in U$ *are such that* $x_\beta \notin E(s_\beta) \cup E(t_\beta) \subseteq E(e_\beta); x, y$ *are two distinct variables such that* $x, y \notin E(u_\alpha) \cup E(v_\alpha)$ *for all* $\alpha \in A$.

(4) $S/\tau \in V_2, S/\mu \in V_1$.

Theorem [14, Theorem 9]. *Let* $W, V_1, V_2 \in \mathcal{L}(CR)$. *If* $V_1 \subseteq \overline{V}_2$ *and* $V_2 \subseteq {V_1}^+$, *then*

$$W \cap (V_1 \vee V_2) = (W \cap V_1) \vee (W \cap V_2) = \overline{W \cap V_2} \cap (W \cap V_1)^+.$$

In [15], the join of the varieties $OLBG$ and $ORBG$ were describeded, and the equality

$$V \cap (OLBG \vee ORBG) = (V \cap OLBG) \vee (V \cap ORBG)$$

was proved to be true. In the following, we generalized the results in [15] by using the operators $\overleftarrow{(\ )}$ and $\overrightarrow{(\ )}$.

Theorem 5.1. *Let* $V_1, V_2 \in \mathcal{L}(CR)$. *Then the equality*

$V_1 \vee V_2 \vee SL = \overrightarrow{V_1} \cap \overleftarrow{V_2}$ *holds if and only if* $V_1 \subseteq \overrightarrow{V_2}, V_2 \subseteq \overleftarrow{V_1}$.

Proof. The direct one is immediate. To prove the converse, assume that $V_1 \subseteq \overrightarrow{V_2}$ and $V_2 \subseteq \overleftarrow{V_1}$. Then we have $V_1 \subseteq \overleftarrow{V_1} \cap \overrightarrow{V_2}, V_2 \subseteq \overleftarrow{V_1} \cap \overrightarrow{V_2}$ and $SL \subseteq \overleftarrow{V_1} \cap \overrightarrow{V_2}$, it follows that $V_1 \vee V_2 \vee SL \subseteq \overleftarrow{V_1} \cap \overrightarrow{V_2}$.

Now we show the opposite inclusion. Let $S \in \overleftarrow{V_1} \cap \overrightarrow{V_2}$. Then by Theorem 3.7 and its dual, we have $S/\tau \cap \mathcal{L}^0 \in V_1 \vee SL$ and $S/\tau \cap \mathcal{R}^0 \in V_2 \vee SL$. Since $\tau \cap \mathcal{L}^0 \cap \mathcal{R}^0 = 1_S$, it folows that S is a subdirect product of $S/\tau \cap \mathcal{L}^0$

and $S/\tau\cap\mathcal{R}^0$, and so $S\in V_1\vee V_2\vee SL$ so that $\overleftarrow{V_1}\cap\overrightarrow{V_2}\subseteq V_1\vee V_2\vee SL$. Now the required conclusion follows. ■

Theorem 5.2. *Let* $V_1,V_2\in\mathcal{L}(CR), SL\subseteq V_1\cap V_2, V_1=[u_\alpha=v_\alpha\colon\ \alpha\in A], V_2=[s_\beta=t_\beta\colon\ \beta\in B]$ *and* $A\cap B=\emptyset$*. If* $V_1\subseteq\overrightarrow{V_2}, V_2\subseteq\overleftarrow{V_1}$*, and* $S\in CR$*. Then the following conditions are equivalent.*

(1) $S\in V_1\vee V_2$.

(2) S *is a subdirect product of a semigroup* $S_1\in V_1$ *and a semigroup* $S_2\in V_2$.

(3) S *satisfies the identities*

$$e_\alpha x_\alpha u_\alpha=e_\alpha x_\alpha v_\alpha,\alpha\in A; s_\beta x_\beta e_\beta=t_\beta x_\beta e_\beta,\beta\in B.$$

Where, for all $\alpha\in A$*, a variable* x_α *and* $e_\alpha\in U$ *are such that* $x_\alpha\notin E(u_\alpha)\cup E(v_\alpha)\subseteq E(e_\alpha)$*; and for all* $\beta\in B$*, a variable* x_β *and* $e_\beta\in U$ *are such that* $x_\beta\notin E(s_\beta)\cup E(t_\beta)\subseteq E(e_\beta)$.

(4) $S/\tau\cap\mathcal{L}^0\in V_1, S/\tau\cap\mathcal{R}^0\in V_2$.

Proof. (1) ⇒ (3). By Theorem 5.1 and by Theorem 3.13 and its dual, the conclusion follows.

(3) ⇒ (4). By Theorem 3.13 and its dual, we have $S\in\overleftarrow{V_1}\cap\overrightarrow{V_1}$. Then by Theorem 3.7 and its dual, we have $S/\tau\cap\mathcal{L}^0\in V_1\vee SL=V_1$ and $S/\tau\cap\mathcal{R}^0\in V_2\vee SL=V_2$, as required.

(4) ⇒ (2). Since $\tau\cap\mathcal{L}^0\cap\mathcal{R}^0=1_S$, it follows that S is a subdirect of $S/\tau\cap\mathcal{L}^0$ and $S/\tau\cap\mathcal{R}^0$. Then the conclusion follows from the hypothesis.

(2) ⇒ (1). It is immediate. ■

Theorem 5.3. *Let* $W,V_1,V_2\in\mathcal{L}(CR)$*, and* $SL\subseteq V_1\cap V_2$*. If* $V_1\subseteq\overrightarrow{V_2}$ *and* $V_2\subseteq\overleftarrow{V_1}$*, then*

$$W\cap(V_1\vee V_2)=(W\cap V_1)\vee(W\cap V_2).$$

Proof. Obviously $W\cap V_1\subseteq W\cap(V_1\vee V_2)$ and $(W\cap V_2)\subseteq W\cap(V_1\vee V_2)$, and thus $(W\cap V_1)\vee(W\cap V_2)\subseteq W\cap(V_1\vee V_2)$.

To show the opposite inclusion, we assume that $S\in W\cap(V_1\vee V_2)$. Then by Theorem 5.2, we have $S/\tau\cap\mathcal{L}^0\in V_1$ and $S/\tau\cap\mathcal{R}^0\in V_2$. So $S/\tau\cap\mathcal{L}^0\in W\cap V_1$ and $S/\tau\cap\mathcal{R}^0\in W\cap V_2$. Since $\tau\cap\mathcal{L}^0\cap\mathcal{R}^0=1_S$, it follows that S is a subdirect product of $S/\tau\cap\mathcal{L}^0$ and $S/\tau\cap\mathcal{R}^0$. Thus $S\in(W\vee V_1)\vee(W\cap V_2)$. The required conclusion now follows. ■

Theorem 5.4. *Under the conditions of Theorem 5.3. If* $SL\subseteq W$*, then*

$$(W\cap V_1)\vee(W\cap V_2)=\overleftarrow{W\cap V_1}\cap\overrightarrow{W\cap V_2}.$$

Proof. By Theorem 3.7 and its dual, it is easily checked that $W\cap V_1\subseteq\overrightarrow{W\cap V_2}$ and $W\cap V_2\subseteq\overleftarrow{W\cap V_1}$. Thus $(W\cap V_1)\vee(W\cap V_2)\subseteq\overleftarrow{W\cap V_1}\cap\overrightarrow{W\cap V_2}$.

Now we show the opposite inclusion. Let $S \in \overleftarrow{W \cap V_1} \cap \overrightarrow{W \cap V_2}$. Then by Theorem 3.7 and its dual and by the facts that $SL \subseteq W \cap V_1$ and $SL \subseteq W \cap V_2$, we have $S/\tau \cap \mathcal{L}^0 \in W \cap V_1$ and $S/\tau \cap \mathcal{R}^0 \in W \cap V_2$. Since $\tau \cap \mathcal{L}^0 \cap \mathcal{R}^0 = l_S$, it follows that S is a subdirect product of $S/\tau \cap \mathcal{L}^0$ and $S/\tau \cap \mathcal{R}^0$. Thus $S \in (W \cap V_1) \vee (W \cap V_2)$. The required conclusion now follws. ■

Theorem 5.5. *Under the conditions of Theorem 5.3. Let* $W \cap V_1 = [u_\alpha = v_\alpha\colon\ \alpha \in A], W \cap V_2 = [s_\beta = t_\beta\colon\ \beta \in B]$ *and* $A \cap B = \emptyset$. *If* $SL \subseteq W \subseteq V_1 \vee V_2$, *then*

$$W = [e_\alpha x_\alpha u_\alpha = e_\alpha x_\alpha v_\alpha, \alpha \in A; s_\beta x_\beta e_\beta = t_\beta x_\beta e_\beta, \beta \in B].$$

Where, for all $\alpha \in A$, *a variable* x_α *and* $e_\alpha \in U$ *are such that* $x_\alpha \notin E(u_\alpha) \cup E(v_\alpha) \subseteq E(e_\alpha)$; *and for all* $\beta \in B$, *a variable* x_β *and* $e_\beta \in U$ *are such that* $x_\beta \notin E(s_\beta) \cup E(t_\beta) \subseteq E(e_\beta)$.

Proof. By Theorem 5.3 and Theorem 5.4, we have $W = (W \cap V_1) \vee (W \cap V_2) = \overleftarrow{W \cap V_1} \cap \overrightarrow{W \cap V_2}$. Now the conclusion follows directly from Theorem 3.13(3) and its dual. ■

Remark 5.6. Let $V_1 = \overleftarrow{V}$ and $V_2 = \overrightarrow{V}$ for some $V \in \mathcal{L}(CR)$. It is easily checked that $V_1 \subseteq \overleftarrow{V_2}, V_2 \subseteq \overrightarrow{V_1}$ and $SL \subseteq V_1 \cap V_2$. So the results in this section can be done on $\overleftarrow{V}$ and $\overrightarrow{V}$. In [15], $OLBG[ORBG]$ was showed to be contained in $\overleftarrow{ORBG}[\overrightarrow{OLBG}]$, and thus the results in this section are generalizations of those in [15].

6. Word problems for free objects in $OLBG$

Since $LBG^+ = LBG$, the word problem for free object in LBG can be solved by using [5, Theorem 4.1]. We try to use the same theorem to solve the word problem for free object in $OLBG$, but we fail to do it [see the following example].

Example. An example for a semigroup in $(OLBG)^+$ but not in $OLBG$. Let S be a Rees Matrix semigroup $M(I, I, G; P)$ where $I = \{1, 2\}, G = \{a, e\}$ with identity e, $P = \begin{pmatrix} e & e \\ e & a \end{pmatrix}$ Then S is a completely simple semigroup on which the Green's relation $\mathcal{H}$ is a congruence, and so $S/\mathcal{H}$ is a rectangular band. Thus $S \in (OLBG)^+$ since $S/\mathcal{H} \in OLBG$. Next we will show that S is not orthodox and therefore $S \notin OLBG$. Clearly $(1, e, 2), (2, e, 1) \in E(S)$, but $(1, e, 2)(2, e, 1) = (1, a, 1) \notin E(S)$, and so S is not orthodox.

However, we can use the operator $\overleftarrow{(\ )}$ to solve the word problem for free object in $OLBG$. We need some lemmas.

Lemma 6.1 [8]. $B \vee G = OBG$.

Lemma 6.2 [15, Theorem 2.3]. $OLBG = [(x^0y)^0 = x^0y^0]$.

Lemma 6.3. $\overleftarrow{OBG} = OLBG$.

Proof. Since $OBG \subseteq OLBG$, $\overleftarrow{OBG} \subseteq \overleftarrow{OLBG}$ by Theorem 2.4. By Lemma 6.2, we have $OLBG = [(x^0y)^0 = x^0y^0]$, then by Theorem 3.13(2), we have $OLBG = \overleftarrow{OLBG}$. Thus $\overleftarrow{OBG} \subseteq OLBG$.

To show the opposite inclusion, assume that $S \in OLBG$, then by [15, Theorem 2.3], we have that $S/\tau \in SLG, S/\mathcal{L}^0 \in B$. Since $S/\tau \cap \mathcal{L}^0$ is a subdirect product of S/τ and $S/\mathcal{L}^0$, it follows that $S/\tau \cap \mathcal{L}^0 \in SLG \vee B$ which is equal to OBG by Lemma 6.1 and by the fact that $G \vee B \subseteq SLG \vee B \subseteq OBG$. Thus $S \in \overleftarrow{OBG}$ by Theorem 3.7. So $OLBG \subseteq \overleftarrow{OBG}$ and $OLBG = \overleftarrow{OBG}$. ■

In order to describe the least orthocryptogroup $(OBG)-$congruence on $U(X)(X = \{x_1, x_2, \ldots\})$, we need a description of the least band congruence β on $U(X)$. The free band is usually given as a quotient of X^+, the free semigroup on X. For any word $w \in U(X)$, let $\overline{w} \in X^+$ be obtained from w by removing all occurrences of (and $)^{-1}$. Since, in a band, $x = x^{-1}$, it is clear that for $w, w' \in U(X), w\beta w'$ if and only if $\overline{w}\overline{\beta}\overline{w'}$, where $\overline{\beta}$ is the least band congruence on X^+. If $u \in X^+$, define $0(u), 1(u)$ and $E(u)$ as before. Then for $u, v \in X^+, u\overline{\beta}v$ if and only if $E(u) = E(v), 0(u)\overline{\beta}0(v)$ and $1(u)\overline{\beta}1(v)$.

Relation π is defined on $U(X)$ in [1] as follows.

Let $w, w' \in U(X)$. Then $w\pi w'$ if and only if $r(w) = r(w')$ and $w\beta w'$.

Lemma 6.4 [1, Theorem 5.2]. *$U(X)/\pi$ is the free orthocryptogroup on X, i.e., for $w, w' \in U(X), w \sim_{OBG} w'$ if and only if $r(w) = r(w')$ and $w\beta w'$.*

Now we can solve the word problem for free object in $OLBG$. We have the following theorem.

Theorem 6.5. *For $w, w' \in U(X).w \sim_{OLBG} w'$ if and only if $E(w) = E(w'), w \sim_{OBG} w', 0(w) \sim_{OLBG} 0(w')$, i.e., $E(w) = E(w'), r(w) = r(w'), w\beta w'$, $0(w) \sim_{OLBG} 0(w')$.*

Proof. By Lemma 6.3, we have that $\sim_{OLBG} = \overleftarrow{\sim_{OBG}}$. Then by the definition of the operator $\overleftarrow{(\ )}$ and by Lemma 6.4, the required conclusion follows. ■

Next we will give an another solution of the word prolbem for free object in $OLBG$, we need some lemmas.

Lemma 6.6. *For $w, w' \in U(X).w \sim_{LRO} w'$ if and only if $E(w) = E(w')$, $r(w) = r(w'), 0(w) \sim_{LRO} 0(w')$.*

Proof. By Theorem 3.8 and by the definition of the operator $\overleftarrow{(\ )}$. ■

Lemma 6.7. $OLBG = B \vee LRO$.

Proof. We have

$$\begin{aligned} OLBG &= \overleftarrow{OBG} \quad \text{by Lemma 6.3} \\ &= \overleftarrow{B \vee G} \quad \text{by Lemma 6.1} \\ &= \overleftarrow{B} \vee \overleftarrow{G} \quad \text{by Theorem 2.4(3)(i)} \\ &= B \vee LRO \quad \text{by Lemma 3.13(2) and Theorem 3.8.} \end{aligned}$$ ■

Theorem 6.8. *For $w, w' \in U(X). w \sim_{OLBG} w'$ if and only if $w \beta w'$ and $w \sim_{LRO} w'$.*

Proof. By Theorem 6.7, we have $\sim_{OLBG} = \sim_B \cap \sim_{LRO}$, then the conclusion follows. ■

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