Tutorial 10: Recurrence and Graphs

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Question

Find an explicit formula for the second-order recurrence relation

$s_0 = 2$, $s_1 = 3$ and $s_k = 2s_{k-1} + 3s_{k-2}$ for $k \geq 2$.

Solution

- Form the auxiliary equation by substituting s_k by t^2, s_{k-1} by t and s_{k-2} by 1:

  $$t^2 = 2t + 3 \qquad \text{or} \qquad t^2 - 2t - 3 = 0$$

- Solve the quadratic equation:

  $$t = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(-3)}}{2} = \frac{2 \pm 4}{2} = -1 \text{ or } 3$$

- Distinct root case.

Solution (cont)

- Solution is of the form $s_k = A \cdot (-1)^k + B \cdot (3)^k$.

  (By the distinct-roots theorem)

- Find A and B from the "initial conditions" s_0 and s_1:

$$
\begin{aligned}
s_0 &= 2 = A + B \\
s_1 &= 3 = -A + 3B
\end{aligned}
$$

- $A = 3/4$, $B = 5/4$.
- So $s_k = \dfrac{3}{4}(-1)^k + \dfrac{5}{4} \cdot 3^k$.
- If $(b^2 - 4ac)$ is negative, you can proceed using complex numbers (must be distinct root in this case).

Question

Find an explicit formula for the second-order recurrence relation

$s_0 = 0$, $s_1 = 4$ and $s_k = 4s_{k-1} - 4s_{k-2}$ for $k \geq 2$.

Solution

- Form the auxiliary equation by substituting s_k by t^2, s_{k-1} by t and s_{k-2} by 1:

  $t^2 = 4t - 4 \qquad \text{or} \qquad t^2 - 4t + 4 = 0$

- Solve the quadratic equation:

  $$t = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(4)}}{2} = \frac{4 \pm 0}{2} = 2$$

- Single root case.

Solution (cont)

- Solution is of the form $s_k = A \cdot (2)^k + B \cdot k \cdot (2)^k$.

  (By the single-root theorem)

- Find A and B from the "initial conditions" s_0 and s_1:

$$\begin{aligned} s_0 &= 0 = A \\ s_1 &= 4 = 2A + 2B \end{aligned}$$

- $A = 0$, $B = 2$.
- So $s_k = 2k \cdot 2^k$.
- (It's just mechanical, nothing special :))

Question

Solve the recurrence relation

$s_0 = 2$, $s_k = 3s_{k-1} + 2$ for $k \geq 1$

(Non-homogeneous because of the term "+2")

Solution

- Transform it into second order linear homogeneous recurrence
- Write the recurrence for s_k and s_{k-1}:

$$\begin{aligned} s_k &= 3s_{k-1} + 2 \\ s_{k-1} &= 3s_{k-2} + 2 \end{aligned}$$

- Subtract the second equation from the first:

  $s_k = 4s_{k-1} - 3s_{k-2}$

- Proceed as usual

Tower of Hanoi with adjacency requirement (Ex. 8.1.18)

Let the pegs are lined up and labelled A, B and C. Now we only allow disk move between adjacent pegs, i.e. direct move from A to C is not allowed. Let a_n be the minimum # moves needed to transfer a tower of n disks from pole A to pole C.

(a) Find a_1, a_2, and a_3.

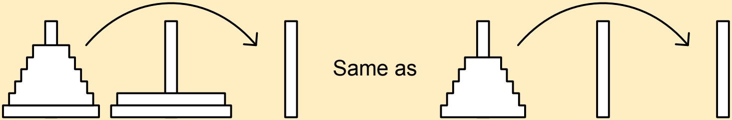
Solution

- $a_1 = 2$. Move disk from A to B and then to C.
- a_2:
  - Move disk 1 from A to C. (2)
  - Move disk 2 from A to B. (1)
  - Move disk 1 back from C to A. (2)
  - Move disk 2 from B to C. (1)
  - Move disk 1 from A to C. (2)

  So $a_2 = 8$.

Solution (cont)

- a_3:
  - Want to move disk 3 from A to B. So need to move things on top to C. ($a_2 = 8$; treat that as a "black box" because all the disks below is of larger size so the "small-on-large" constraint is not violated.)



  - Move disk 3 from A to B. (1)
  - Now want to move disk 3 from B to C. So need to move the 2 disks from C to A. (8; swap A and C in the above procedure)
  - Move disk 3 from B to C. (1)
  - Move the 2 disks from A to C. (8)
- $a_3 = 26$. Generalize it?

Tower of Hanoi with adjacency requirement (Ex. 8.1.18)

(c) Find a recurrence relation for a_k

Solution

- a_k: (Let T_k be the tower of the top k disks)
  - Want to move disk $\cancel{3}$ k from A to B. So need to move ~~the 2 disks~~ T_{k-1} to C. (a_{k-1} moves; treat transferring of T_{k-1} as a "black box")
  - Move disk k from A to B. (1)
  - Now want to move disk k from B to C. So need to move T_{k-1} from C to A. (a_{k-1} moves, by symmetry.)
  - Move disk k from B to C. (1)
  - Move T_{k-1} from A to C. (a_{k-1})
- So we have recurrence relation for a_k:

  $a_1 = 2$, $a_k = 3a_{k-1} + 2$ for $k \geq 2$.

Question

Is this optimal?

Solution

Yes. For the sake of moving the bottom disk, everything on top of it needs to be relocated elsewhere, which there is only one option (the pole other than the original pole and the destination pole)

Question

Is there a non-recursive solution for the original puzzle, which is easier to do by hand?

Solution

Yes. See http://en.wikipedia.org/wiki/Tower_of_Hanoi#Non-recursive_solution for details.

Question

What if there are 4 or more poles?

Solution

There are similar methods to solve recursively, but not proven to be optimal. See http://en.wikipedia.org/wiki/Tower_of_Hanoi#Four_pegs_and_beyond for details.

Suggestion

Try out other variations of the Tower of Hanoi

- 8.1.19 – 8.1.21 of textbook.
- What if we can only move disks from A to B, B to C, C to A, but not the other way round?

T/F Question 1

A graph without parallel edges is simple.

Solution

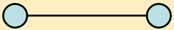
- A simple graph should have no parallel edges...

- ... as well as no loops!
- False

T/F Question 2

$(2, 2, 1, 1, 1)$ is a degree sequence of a graph.

Solution

- False. Since the sum of degree is odd number, it cannot be a degree sequence of any graph
- By handshaking lemma, sum of degrees = twice # edges...
- ... because every edge has two end points. 
- Implication: A graph cannot have an odd number of vertices with odd degree, e.g. 3 degree-1 vertices in the problem

Definition

Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there is an *edge-preserving* vertex mapping, i.e.

$$\exists \textit{ bijection } f : V_1 \to V_2 \text{ s.t.}$$
$$u—v \text{ in } E_1 \;\; \text{if and only if} \;\; f(u)—f(v) \text{ in } E_2$$

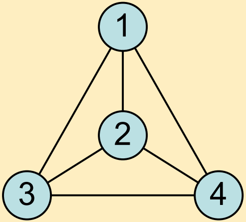
Example

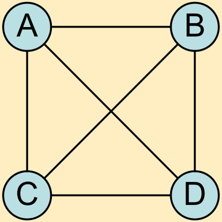
Graphs that are isomorphic has

- Same # vertices
- Same # edges
- Same degree sequence
- Same properties (bipartiteness, connectedness, . . .)

Question

Are the two graphs isomorphic to each other?





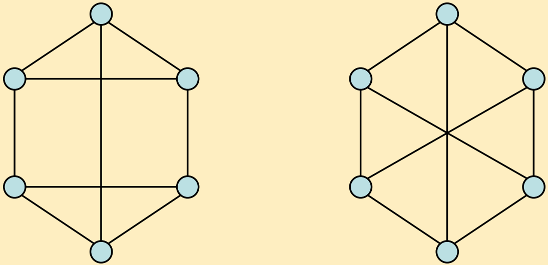
Solution

- Map 1 to A, 2 to B, 3 to C and 4 to D
- Yes. (In fact any mapping will do)
- Showing isomorphism is easy: Just show him the mapping of vertices. He can verify by checking all pairs of vertices by himself: $n(n-1)/2$ pairs

Question

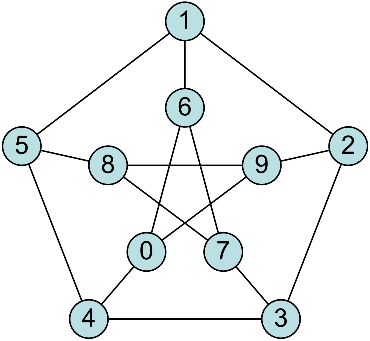
Same degree sequence imply the two graphs are isomorphic.

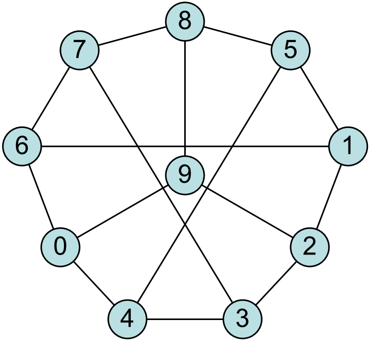
Solution



- Both are $(3, 3, 3, 3, 3, 3)$ but NOT isomorphic
- Reason: Triangle on the left but not on the right
- (Give a degree sequence for a simple graph that is unique)

Showing isomorphism is easy; finding the bijection is not. . .

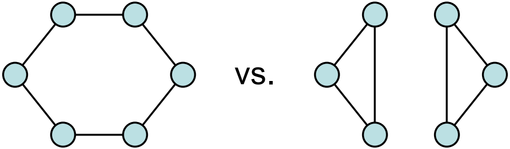




(Give a degree sequence for a simple graph that yields unique graph up to isomorphism)

There are a lot of solutions, including:

- $(0, 0, 0, \ldots)$: Graph with no edges
- $(1, 1, 1, \ldots)$ (n even): Vertices are paired up into pairs
- $(n-1, n-1, n-1, \ldots)$: Complete graph (there is an edge between every pair of vertices)
- $(n-2, n-2, n-2, \ldots)$ (n even): Consider the complement (edge $\rightarrow$ non-edge and vice versa): becomes $(1, 1, 1, \ldots)$. (Two graphs are isomorphic if and only if their complement are isomorphic)
- How about $(2, 2, 2, \ldots)$?

- 

  $\Rightarrow$ NO! There can be disjoint cycles

- $(1, n-2, n-2, \ldots, n-2, n-1)$: